ELEC 533 Homework 6

Due date: In class on Wednesday, October 25th, 2000

Instructor: Dr. Rudolf Riedi

23. With fixed λ, for each integer $n \geq \lambda$, let $X_{1,n}$, $X_{2,n}$,..., $X_{n,n}$ be independent random variables such that

$$P[X_{i,n} = 1] = \frac{\lambda}{n}$$
$$P[X_{i,n} = 0] = 1 - \frac{\lambda}{n}.$$

Let $Y_n = X_{1,n} + X_{2,n} + ... + X_{n,n}$.

(a) Find Φ_{Y_n}, the characteristic function of Y_n.

(b) One can interpret Y_n as the number of successes in n independent Bernoulli trials with success probability λ/n. Verify that Y_n has a binomial distribution! Compute $\mathbb{E}[Y_n]$!

(c) Find the limit of Φ_{Y_n} as $n \to \infty$. What distribution does it correspond to? How does your finding relate to an 'old' result from class?

24. Consider the experiment of an infinite series of independent Bernoulli trials with success probability p and let

$$X_n = \begin{cases} 1 & \text{if the } n\text{-th experiment is successful} \\ 0 & \text{if the } n\text{-th experiment fails.} \end{cases}$$

More precisely, this means that the universal set Ω is the set of all 0-1-sequences: $\Omega = \{0,1\}^{\mathbb{N}}$ and the probability on Ω is defined by the probabilities

$$P[\{\omega = (\omega_1, \omega_2, \ldots) : \omega_n = \text{success}\}] = P[X_n = 1] = p$$

for all fixed n. Define $S_m(\omega) := \sum_{n=1}^{m} X_n(\omega) \cdot 2^{-n}$.

(a) Show that $S_m(\omega)$ converges as $m \to \infty$ for all ω. It follows that S_m converges 'almost surely'. Let us denote its limit by $T(\omega)$. (This is again a r.v.)

(b) Show that S_m converges in the mean square sense, i.e. show that $\mathbb{E}[|S_m - T|^2]$ converges to zero as $m \to \infty$.

(c) Let us compute F_T for the special case $p = 1/2$.
Start by noting that the smallest possible value of T is 0, and the largest is 1 and conclude that $F_T(a) = 0$ for $a \leq 0$ and $F_T(a) = 1$ for $a \geq 1$.
Now, show that for all integer m and all $k = 0, \ldots 2^m - 1$

$$P[k2^{-m} \leq T < (k+1)2^{-m}] = 2^{-m}.$$

Finally, conclude that T is uniformly distributed in the interval $[0, 1]$.

(d) Describe T and F_T in the cases $p = 0$ and $p = 1$.

Remark: When $p \notin \{0, 1/2, 1\}$, then F_T is a continuous function but has no density, i.e. there is no function f such that $F_T(a) = \int_0^a f(t)dt$. Such CDF-s are called singular distributions. The one given in this example is the simplest case of a multifractal.