ELEC 533 Homework 8

Due date: In class on Wednesday, November 8th, 2000

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28. Let X_n be a sequence of independent, identically distributed (iid) symmetrical stable variables, more precisely $X_n \simeq S\alpha S(\mu,\sigma)$ $(0 < \alpha \leq 2)$, which means that

$$\Phi_{X_n}(u) = e^{(i\mu u - |\sigma|^\alpha |u|^\alpha)},$$

where μ is the position parameter, and σ is the scale parameter.

(a) Using the rules for the characteristic function show that the sum of two *independent* $S\alpha S$ r.v.'s is also $S\alpha S$. (Note that a Cauchy r.v. is $S\alpha S$ with $\alpha = 1$: so, we solved this problem for the special case $\alpha = 1$ already earlier.)

(b) Let $X \simeq S\alpha S(\mu,\sigma)$ and assume that $\alpha > 1$. Show that $\mu = \mathbb{E}[X]$. (Note: When $\alpha \leq 1$, then $\mathbb{E}[|X|] = \infty$. Hence $\mathbb{E}[X]$ is not defined when $\alpha \leq 1$.)

(c) For the remainder of the homework assume that $\mu = 0$. Use the characteristic function to show that

$$Z_n := \frac{X_1 + \ldots + X_n}{n^{1/\alpha}}$$

are distributed as X_n, i.e. $Z_n \simeq S\alpha S(0,\sigma)$. Conclude that Z_n converges in distribution. (The lesson to learn is that for the sum of Stable random variables one has to choose carefully the renormalization factor to obtain convergence in distribution to again a stable law, and this factor it is not the $1/\sqrt{n}$, the one used in the CLT (unless $\alpha = 2$).)

(d) Recall that Cauchy variables are the special case of $S\alpha S$ variables with $\alpha = 1$. Use (28c) to show that the Law of Large Numbers (LLN) does not hold for a sequence of iid Cauchy random variables. Which assumption(s) of the LLN stated in class are violated?

(e) Still using (28c) show that the Central Limit Theorem does not hold for a sequence X_n of iid $S\alpha S$ random variables $(\alpha < 2)$. Which assumption(s) of the CLT stated in class are violated?

(f) Still for the sequence X_n of (iid) $S\alpha S$ random variables: Guess for which values of the parameter α does the weak LLN hold? Give a heuristic answer only. No reasoning required.