ELEC 533 Homework 2

Due date: In class on Friday, September 21

Instructor: Dr. Rudolf Riedi

6. Simplified Roulette (without "zero"): $\Omega = \{1, \ldots, 36\}$, $P[\{n\}] = 1/36$ for all $n \in \Omega$. There are 3 events we are interested in: E are the even numbers, R are the red numbers, and F are the numbers in the first row, i.e. $F = \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34\}$. Unlike true roulette let us assume that the red numbers are $R = \{1, 2, 3, 7, 8, 12, 13, 14, 15, 19, 20, 24, 25, 26, 27, 31, 32, 36\}$:

1	4	**7**	10	**13**	16	**19**	22	**25**	28	**31**	34
2	5	**8**	11	**14**	17	**20**	23	**26**	29	**32**	35
3	6	9	**12**	**15**	18	21	**24**	**27**	30	33	**36**

   (a) Compute $P[E \cap R]$, $P[E \cap F]$, $P[R \cap F]$, $P[E \cap R \cap F]$.

   (b) Compute $P[E \,|\, R]$, $P[E \,|\, F]$, $P[R \,|\, F]$.

   (c) Are the events E and R independent?

   (d) Are the events E and F independent?

   (e) Are the events R and F independent?

   (f) Are the events E, R and F independent?

   (g) Let us assume that instead of the first row, $F = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$ denotes the numbers in the *third* row. Of all the 11 questions in 6a to 6f, exactly two have now a different answer. Which are they, and what are the new answers?

7. Compute the distribution functions —i.e., $F(a) = P[(-\infty, a])$ for *all* $a \in \mathbb{R}$ — for the following distributions. For (7b) verify also, that this is indeed a valid probability law. In all but the last one, the laws are given through a density.

   (a) $\mathcal{U}([0, 2\pi])$: Uniform on $[0, 2\pi]$

$$f(x) = \begin{cases} \frac{1}{2\pi} & \text{for } x \text{ in } [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Cauchy:

$$f(x) = \frac{1}{\pi(1 + x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

   (c) $\exp(\lambda)$: One sided exponential with parameter $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (d) The probability law induced by the following random variable which is maps on $\Omega = \{\texttt{head}, \texttt{tail}\}$ with $P[\{\texttt{head}\}] = 1/3$. The random variable is defined as $X(\texttt{head}) = 3$, $X(\texttt{tail}) = -6$.

8. In a communication system with input X and output Y, a zero or one is transmitted with $P[X = 0] = P_0$, $P[X = 1] = 1 - P_0 = P_1$ respectively. Due to noise in the channel, a zero can be detected as a one with probability β and a one can be detected as a zero also with probability β (i.e $P[Y = 0|X = 1] = P[Y = 1|X = 0] = \beta$). The receiver detects a one. What is the probability that a one was transmitted?

9. Suppose $X \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. X is a r.v. uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Let $Y = \tan(X)$. The distribution of Y can be computed in two different ways.

   (a) Use $P[Y \leq y] = P[\tan(X) \leq y]$ to show that $F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\arctan(y)$. Then infer the density $f_Y(y)$ of the distribution of Y as the derivative of $F_Y(y)$ and conclude that Y is a Cauchy r.v.

   (b) The following formula connecting the densities of the distributions of the random variables X and $Y = g(X)$ is valid if g is differentiable and strictly increasing:

   $$f_Y(g(x)) = \frac{f_X(x)}{g'(x)}, \qquad \text{or} \qquad f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y).$$

   Use it to derive $f_Y(y)$ for $g(X) = \tan(X)$.

   This problem provides an intuitive interpretation of the Cauchy law: A person is blindfolded, standing in front of a wall. The person spins around a few times. The angle under which the person now "faces" the wall is completely arbitrary, in other words, uniformly distributed. The person starts walking straight. The location where the person will collide with the wall is random, with Cauchy distribution.

10. Suppose that X (signal) is a binary r.v. with $P[X = 1] = \alpha$ and $P[X = -1] = \beta = 1 - \alpha$. Suppose the r.v. N is normal, i.e. $N \sim \mathcal{N}(0, \sigma^2)$ (noise). Assume also that N and X are independent, meaning that the events $\{X = a\}$ and $\{N \leq b\}$ are independent for all a and b. We are interested in the r.v. $Y = X + N$ (noisy observation).

    (a) Express $P[Y \leq y|X = 1]$ and $P[Y \leq y|X = -1]$ in terms of Gaussian integrals. (Note that $\int_{-\infty}^{t} \exp(-x^2)dx$ has no closed form.)

    (b) Using this and the law of total probability find $F_Y(y)$, again in terms of integrals.

    (c) Derive a closed form expression for f_Y from F_Y. This is a mixture density; identify it's components in terms of known density functions.

    (d) To infer the signal X from the observation Y the following strategy is used:

    - If $Y \geq \gamma$, we infer that $X = 1$,
    - if $Y < \gamma$, we infer that $X = -1$,

    where γ is some number we will decide on later (next question). We make an error in the inference if $Y \geq \gamma$ and $X = -1$, or if $Y < \gamma$ and $X = 1$. Compute the probability p_{e} that this happens.

    (e) Find the γ that minimizes the probability of error p_{e}.