21. Let \( X \sim \mathcal{N}(\mu, \sigma^2) \) be a Gaussian r.v. and set \( Y = e^X \). \( Y \) is called log-normal.

(a) Compute \( \mathbb{E}[Y] \) in terms of \( \mu \) and \( \sigma \).
(b) Use above formula to compute \( \mathbb{E}[Y^q] \) for any \( q \in \mathbb{R} \).
(c) Use this to compute Var\( (Y) \).
(d) Verify that for a Gaussian r.v. \( X \) we have \( \mathbb{E}[e^X] > e^{\mathbb{E}[X]} \).

22. (a) Let \( X \sim \text{Pois}(\lambda) \) (see Homework problem 7 (a)). Compute the characteristic function \( \Phi_X(u) \) of \( X \).
(b) Let \( X_1 \sim \text{Pois}(\lambda_1) \) and \( X_2 \sim \text{Pois}(\lambda_2) \) be independent r.v.’s. Show that \( X_1 + X_2 \sim \text{Pois}(\lambda_1 + \lambda_2) \).
Thus, the sum of two independent Poisson r.v.’s is also Poisson. (HINT: Use the characteristic function.)
(c) Let \( X \sim \exp(\lambda) \) (see Homework problem 7 (c)). Compute the characteristic function \( \Phi_X(u) \) of \( X \).
(d) Is the sum of two independent exponential r.v.’s an exponential r.v.?

23. (a) Let \( X \sim \mathcal{C}(0, 1) \): a standard Cauchy r.v. which is given through the density
\[
f_X(x) = \frac{1}{\pi(1 + x^2)} \quad \text{for every } x \in \mathbb{R}.
\]
Verify that the characteristic function \( \Phi_X(u) \) is given by
\[
\Phi_X(u) = e^{-|u|}.
\]
(b) Let \( Y \sim \mathcal{C}(a, b) \) be a general Cauchy r.v., meaning that \( Y = a + bX \), where \( X \sim \mathcal{C}(0, 1) \) and where \( a \) and \( b \) are constants. Compute the characteristic function \( \Phi_Y \) of \( Y \). (Pay attention to the sign of \( b \), i.e. \( \Phi_X(u) = \Phi_{-X}(u) = \Phi_X(-u) \) by symmetry.)
(c) Let \( Y_1 \sim \mathcal{C}(a_1, b_1) \) and \( Y_2 \sim \mathcal{C}(a_2, b_2) \) be independent Cauchy r.v.’s. Show that the sum \( Y := Y_1 + Y_2 \) is also Cauchy: \( Y \sim \mathcal{C}(a, b) \). Compute \( a \) and \( b \) from \( a_1, a_2, b_1 \) and \( b_2 \).

24. With fixed \( \lambda \), for each integer \( n \geq \lambda \), let \( X_{1,n}, X_{2,n}, \ldots, X_{n,n} \) be independent random variables such that
\[
P[X_{i,n} = 1] = \frac{\lambda}{n}
\]
\[
P[X_{i,n} = 0] = 1 - \frac{\lambda}{n}.
\]
Let \( Y_n = X_{1,n} + X_{2,n} + \ldots + X_{n,n} \).

(a) Find \( \Phi_{Y_n} \), the characteristic function of \( Y_n \).
(b) One can interpret \( Y_n \) as the number of successes in \( n \) independent Bernoulli trials with success probability \( \lambda/n \). Verify that \( Y_n \) has a binomial distribution! Compute \( \mathbb{E}[Y_n] \)!
(c) Find the limit of \( \Phi_{Y_n} \) as \( n \to \infty \). What distribution does it correspond to? How does your finding relate to an ‘old’ result from class?

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\(^1\)This notation is not common and should only be used in this homework set.