ELEC 533 Homework 9

Due date: Friday, November 30, 2001

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32. Let $\{N_t, t \in \mathbb{R}\}$ be a Poisson process with intensity λ.

    (a) Compute the second order joint statistics $F_{N_t N_s}$, i.e. compute $P[N_t = n, N_s = m]$.
        Hint: Increments.

33. Suppose X_i ($i \in \mathbb{N}$) is a sequence of iid random variables and suppose

    $$P[X_i = 1] = P[X_i = -1] = 1/2$$

    for all i. Define a random process $\{Y_t, t \in \mathbb{R}\}$ by

    $$Y_t = X_i \qquad \text{for all } t \text{ such that } i - 1 < t \leq i.$$

    (a) Sketch a typical sample path of $\{Y_t, t \in \mathbb{R}\}$.

    (b) Find the mean and autocorrelation function of $\{Y_t, t \in \mathbb{R}\}$. Is the process wide sense stationary
        (wss)? Show your argument.

    (c) Suppose A is a r.v. independent of the X_i and uniformly distributed in $[0,1]$: $A \sim \mathcal{U}([0,1])$.
        Define the random process $\{Y_t, t \in \mathbb{R}\}$ by

        $$Z_t = Y_{t+A}$$

        Sketch a typical sample path of $\{Z_t, t \in \mathbb{R}\}$.

    (d) Is Z_t wss? Is Z_t second order stationary? Why or why not?

34. Compute the spectral density for processes with the following auto-correlation:

    (a) $R_X(\tau) = \exp(-\beta|\tau|)$.

    (b) $R_X(\tau) = \max(1 - |\tau|/T, 0)$. HINT: R_X is the convolution of two extremely simple functions
        with extremely simple Fourier transforms.

    The first example is the auto-correlation of a Gauss-Markov process, the second can be obtained running
    white noise through an integral averager (see class).

35. Let $\{X_t\}$ be a continuous time, but discrete valued Markov process. For simplicity we assume that
    the values are integers. As in the case with continuous values treated in class, we define the process
    via its initial distribution $P[X_0 = k]$ at time 0 and the transition probabilities

    $$P[X_t = k | X_s = n]$$

    for any $t > s$. More precisely, the marginals are

    $$P[X_t = k] = \sum_{n=1}^{\infty} P[X_t = k | X_0 = n] \cdot P[X_0 = n]$$

    and the f.d.d. are

    $$P[X_{t_1} = k_1, \ldots X_{t_n} = k_n] = P[X_{t_n} = k_n | X_{t_{n-1}} = k_{n-1}] \cdots P[X_{t_2} = k_2 | X_{t_1} = k_1] \cdot P[X_{t_1} = k_1]$$

    As in the continuous-valued case done in class, this defines a consistent f.d.d. if and only if the
    Chapman-Kolmogorov equations are satisfied. Here, these Chapman-Kolmogorov equations read
    as:

    $$\sum_{n=1}^{\infty} P[X_t = k | X_s = n] \cdot P[X_s = n | X_r = m] = P[X_t = k | X_r = m].$$

(a) Check that the Chapman-Kolmogorov equations imply consistency. To this end, assume that the Chapman-Kolmogorov equations hold and show that for any $j = 1, \ldots n$

$$\sum_{k_j=1}^{\infty} P[X_{t_1} = k_1, \ldots X_{t_n} = k_n] = P[X_{t_1} = k_1, \ldots, X_{t_{j-1}} = k_{j-1}, X_{t_{j+1}} = k_{j+1}, \ldots, X_{t_n} = k_n].$$

(b) Consider the following particular transition probability where $t > s$ and where k and n are positive integers:

$$P[X_t = k | X_s = n] = \begin{cases} 0 & \text{if } k < n \\ e^{-\lambda(t-s)}(\lambda(t-s))^{k-n}/(k-n)! & \text{if } k \geq n. \end{cases}$$

In other words, given that $X_s = n$ the *increment* $X_t - X_s$ is a Poisson random variable of mean $\lambda(t-s)$.

Verify that the Chapman-Kolmogorov equations are satisfied. Hint 1: recall that

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}.$$

Hint 2: be careful about noting when $P[X_t = k | X_s = n]$ is zero.

(c) With the same transition probabilities as before add the initial distribution $P[X_0 = 0] = 1$. Show that X_t is a Poisson variable of mean λt.

Note: In this case, it is possible to show that X_t is actually a Poisson renewal process. This is not obvious, since one needs to construct arrival times T_k of events which should be spaced by independent waiting times. A first step towards this goal would be to show that the increments $X_t - X_s$ over disjoint time intervals are independent.

Here are some more problems to practice your skill. They are not mandatory.

36. Suppose A and B are independent Gaussian random variables, i.e. $A \sim \mathcal{N}(0,1)$ and $B \sim \mathcal{N}(0,1)$. Define the random process $\{X_t, t \in \mathbb{R}\}$ by

$$X_t = A + Bt + t^2.$$

(a) Compute the following marginal distributions: ϕ_{X_t} for arbitrary t and ϕ_{X_1, X_5}.

(b) Are $X_1, X_2, \ldots, X_5$ jointly Gaussian?

(c) Find $\mathbb{E}[X_5 \mid X_0]$. This is a prediction of X_5 knowing X_0, and it is a random variable. Compute its variance.

(d) Find $\mathbb{E}[X_5 \mid X_0, X_1]$. This is a prediction of X_5 knowing X_0 and X_1, and it is also a random variable. Compute its variance.

37. Throughout let all processes be wss. Recall: for X_t the power spectral density $S_X(\nu)$ is defined as the Fourier transform of the auto-correlation function $R_X(\tau) = \mathbb{E}[X_{s+\tau} \cdot X_s^*]$.

(a) Let U_t and V_t be orthogonal wss processes, meaning that $\mathbb{E}[U_t V_s] = 0$ for all t and s. Set $W_t = U_t + V_t$. Show that $R_W(\tau) = R_U(\tau) + R_V(\tau)$ and conclude that $S_W(\nu) = S_U(\tau) + S_V(\tau)$.

(b) If $Y_t = aX_t$ then $R_Y(\tau) = |a|^2 R_X(\tau)$ and $S_Y(\nu) = |a|^2 S_X(\nu)$.

(c) Let $Y_t = X_t \cdot \cos(\nu_0 t + \Theta)$ where ν_0 is a fixed frequency and where the random phase Θ is independent of the process X_t and uniformly distributed on $[-\pi, \pi]$. Show that $R_Y(\tau) = R_X(\tau)\cos(\nu_0\tau)$ and $S_Y(\nu) = 1/2(S_X(\nu + \nu_0) + S_X(\nu - \nu_0))$.

(d) Assume that X_t be a zero mean process, meaning that $\mu_X(t) = 0$. Set $Z_t = X_t + b$. Show that $R_Z(\tau) = R_X(\tau) + |b|^2$ and $S_Z(\nu) = S_X(\nu) + 2\pi|b|^2\delta(\nu)$.

(e) Assume that A_t is a wss process such that its auto-correlation function is related to that of X_t via $R_A(\tau) = -R_X''(\tau)$. Show that $S_A(\nu) = \nu^2 S_X(\nu)$. (Here, R'' denotes the second derivative.)

Most of these questions can be derived from properties of the Fourier transform.