ELEC 533 Homework 7

Due date: In class on Friday, October 25th, 2002

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26. Let X_n be Gaussian r.v.-s with mean μ_n and variance σ_n^2. Under what conditions on the sequences μ_n and σ_n^2 does X_n converge in distribution and what is the limiting distribution.

27. There is a theorem which states that if there is a random variable Y with $E[|Y|^2] < \infty$ such that $|X_m| \leq Y$, almost surely, and if X_m converges in probability then X_m converges also in the mean square sense. Your task is to prove a simplified version of this fact which goes as follows:
Suppose that
$$X_m \xrightarrow{\text{i.p.}} X$$
and suppose that there is a constant c such that $|X_m| \leq c$ for all n, then
$$X_m \xrightarrow{\text{m.s.}} X.$$
[HINT: You need to show that $\mathbb{E}[|X_m - X|^2] \to 0$. To this end, break this expectation into two terms using the law of total probability:
$$\mathbb{E}[|X_m - X|^2 \,\Big|\, |X_m - X| > \varepsilon] \cdot P[|X_m - X| > \varepsilon] + \mathbb{E}[|X_m - X|^2 \,\Big|\, |X_m - X| \leq \varepsilon] \cdot P[|X_m - X| \leq \varepsilon].$$
Here $\varepsilon > 0$ is any positive number. You want to show that both terms can be made arbitrarily small. For the first term use the boundedness of X_m together with the fact that X_m converges in probability. For the second term use that a r.v. bounded by a small number has a small second moment.]

28. Suppose that
$$X_m \xrightarrow{\text{D}} X$$
and that there is a constant c such that $P[X = c] = 1$. Show that
$$X_m \xrightarrow{\text{i.p.}} X$$
[HINT: You need to show that $P[|X_m - X| > \varepsilon] \to 0$ $(m \to \infty)$ for any ε. Because X is essentially constant and equal to c, this probability is equal to $1 - P[c - \varepsilon \leq X_m \leq c + \varepsilon]$ which is easily expressed in terms of the CDFs F_{X_m}. Now, use that F_{X_m} converges to F_X, which has a particularly simple form because X is essentially constant.]

29. Let X be a Binomial random variable which takes the values 1 and -1 both with probability 1/2.
    (a) Compute the characteristic function of X, i.e., $\phi_X(u) = \mathbb{E}[\exp(iuX)]$. Find a simple expression in terms of a cosine function.
    (b) Verify that $\phi''(0) = -\mathbb{E}[X^2]$.
    (c) Compute the characteristic function of
    $$Y_n = \frac{X_1 + \ldots + X_n}{2\sqrt{n}}$$
    using the above simple formula. Here, the random variables X_n are independent and of the same distribution as X.
    (d) Approximate the cosine function by its Taylor polynomial of order 2 (i.e., the quadratic polynomial that provides the best approximation) and compute the limit of the characteristic function of Y_n. Conclude that Y_n converges in distribution. What is the limiting distribution?