ELEC 533 Homework 2

Due date: In class on Wednesday, September 17

Instructor: Dr. Rudolf Riedi

6. Consider simplified Roulette (without "zero"): $\Omega = \{1, \ldots, 36\}$, $P[\{n\}] = 1/36$ for all $n \in \Omega$. There are 3 events we are interested in: E are the even numbers, R are the red numbers, and F are the numbers in the first row, i.e. $F = \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34\}$. Unlike true roulette let us assume that the red numbers are $R = \{1, 2, 3, 7, 8, 12, 13, 14, 15, 19, 20, 24, 25, 26, 27, 31, 32, 36\}$:

1	4	**7**	10	**13**	16	**19**	22	**25**	28	**31**	34
2	5	**8**	11	**14**	17	**20**	23	**26**	29	**32**	35
3	6	9	**12**	**15**	18	21	**24**	**27**	30	33	**36**

   (a) Compute $P[E \cap R]$, $P[E \cap F]$, $P[R \cap F]$, $P[E \cap R \cap F]$.
   (b) Compute $P[E \,|\, R]$, $P[E \,|\, F]$, $P[R \,|\, F]$.
   (c) Are the events E and R independent?
   (d) Are the events E and F independent?
   (e) Are the events R and F independent?
   (f) Are the events E, R and F independent?
   (g) Let us assume that instead of the first row, $F = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$ denotes the numbers in the *third* row. Of all the 11 questions in 6a to 6f, exactly two have now a different answer. Which are they, and what are the new answers?

7. Compute the distribution functions —i.e., $F(a) = P[(-\infty, a])$ for *all* $a \in \mathbb{R}$ — for the following distributions. For (7b) verify also, that this is indeed a valid probability law. In all but the last one, the laws are given through a density.

   (a) $\mathcal{U}([0, 2\pi])$: Uniform on $[0, 2\pi]$

$$f(x) = \begin{cases} \frac{1}{2\pi} & \text{for } x \text{ in } [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Cauchy:

$$f(x) = \frac{1}{\pi(1 + x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

   (c) $\exp(\lambda)$: One sided exponential with parameter $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (d) Given is a probility space $\Omega = \{\texttt{head}, \texttt{tail}\}$ with $P[\{\texttt{head}\}] = 1/3$. Given is also a random variable defined as $X(\texttt{head}) = 3$, $X(\texttt{tail}) = -6$. What is the probability law (distribution) on the real line induced by the random variable X? More precisely, compute $P_{\mathbb{R}}[(-\infty, a]] = P_\Omega[X \leq a]$ for all a.

(see second page)

8. A large class in probability theory is taking a multiple choice test. For one particular question with m proposed multiple choice answers, the fraction of students who know the answer is p; the others will guess. The probability of answering the question correctly is 1 for the students who know the answer and $1/m$ for the ones who guess. What is the probability that a student knows the answer given that she or he has answered it correctly.

9. Suppose $X \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. X is a r.v. uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Let $Y = \tan(X)$. The distribution of Y can be computed in two different ways.

   (a) Use $P[Y \leq y] = P[\tan(X) \leq y]$ to show that $F_Y(y) = \frac{1}{2} + \frac{1}{\pi} \arctan(y)$. Then infer the density $f_Y(y)$ of the distribution of Y as the derivative of $F_Y(y)$ and conclude that Y is a Cauchy r.v.

   (b) The following formula connecting the densities of the distributions of the random variables X and $Y = g(X)$ is valid if g is differentiable and strictly increasing:

   $$f_Y(g(x)) = \frac{f_X(x)}{g'(x)}, \qquad \text{or} \qquad f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y).$$

   Use it to derive $f_Y(y)$ for $g(X) = \tan(X)$.

   This problem provides an intuitive interpretation of the Cauchy law: A person is blindfolded, standing in front of a wall. The person spins around a few times. The angle under which the person now "faces" the wall is completely arbitrary, in other words, uniformly distributed. The person starts walking straight. The location where the person will collide with the wall is random, with Cauchy distribution.