ELEC 533 Homework 4

Due date: In class on Wednesday, October 8, 2003

Instructor: Dr. Rudolf Riedi

14. Assume that X and Y are independent Gaussian random variables with zero mean and variance 1. Compute the distribution of the random variable $Z = \exp(-(X^2+Y^2)/2)$. Hint: the transformation from Cartesian to polar coordinates goes as $x = r\sin(\phi)$, $y = r\cos(\phi)$, $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\phi$.

15. Suppose X and Y are joint r.v.'s with

$$f_{XY}(x,y) = \begin{cases} e^{-y} & x > 0, y > x \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $\iint f_{XY}(x,y)\, dy\, dx = 1$.
   (b) Find f_X and f_Y.
   (c) Are X and Y independent? Show your reasoning.

16. Let the joint density of the pair of random variables (X,Y) be given by

$$f_{XY}(x,y) = \begin{cases} y\exp(-xy) & \text{if } x > 1 \text{ and } y > 0 \\ 0 & \text{else.} \end{cases}$$

   (a) Compute the marginal densities f_X and f_Y.
   (b) Are X and Y independent? Show your reasoning.
   (c) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

17. Two pairs of discrete random variables (U,V) and (X,Y) are given via their joint distributions:

$$P[U=u, V=v] = \begin{cases} 1/2 & \text{if } u = 1,\ v = 0 \\ 1/6 & \text{if } u = 1,\ v = 1 \\ 1/12 & \text{if } u = -1,\ v = 1 \\ 1/4 & \text{if } u = -1,\ v = 0 \end{cases}$$

and

$$P[X=x, Y=y] = \begin{cases} 7/12 & \text{if } x = 1,\ y = 0 \\ 1/12 & \text{if } x = 1,\ y = 1 \\ 1/6 & \text{if } x = -1,\ y = 1 \\ 1/6 & \text{if } x = -1,\ y = 0 \end{cases}$$

   (a) Show that the marginals are the same, that is $F_X = F_U$ and $F_Y = F_V$.
   (b) Which pair is independent?

18. Let X and Y be standard jointly Gaussian r.v.'s ($\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$) with joint density

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right),$$

where ρ is a constant.

   (a) Show by direct computation that $\mathrm{cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = \rho$.
   (b) Compute the density f_Z of the sum $Z = X + Y$ using the general formula for the density of the sum of (dependent) random variables: $f_Z(z) = \int f_{XY}(x, z-x)dx$. Conclude that Z is as well Gaussian.