ELEC 533 Homework 6

Due date: In class on Wednesday, October 22, 2003

Instructor: Dr. Rudolf Riedi

23. Let $X \simeq \mathcal{N}(\mu, \sigma^2)$ be a Gaussian r.v. and set $Y = e^X$. Y is called log-normal.
    (a) Compute $\mathbb{E}[Y]$ in terms of μ and σ.
    (b) Compute $\mathbb{E}[Y^q]$ for any $q \in \mathbb{R}$. Hint: Write $Y^q = (e^X)^q = e^{(qX)}$ and use (a).
    (c) Use this to compute $\mathrm{Var}(Y)$.
    (d) Jensen's inequality says that $\mathbb{E}[\exp(X)] > \exp(\mathbb{E}[X])$. Check this fact by explicit comparison of $\mathbb{E}[Y]$ and $\mathbb{E}[X]$.

24. (a) Let $X \simeq \mathcal{C}(0,1)$:[1] a standard Cauchy r.v. which is given through the density

$$f_X(x) = \frac{1}{\pi(1+x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

    Verify that the characteristic function $\Phi_X(u)$ is given by

$$\Phi_X(u) = e^{-|u|}.$$

    (b) Let $Y \simeq \mathcal{C}(a,b)$ be a general Cauchy r.v., meaning that $Y = a + bX$, where $X \simeq \mathcal{C}(0,1)$ and where a and b are constants. Compute the characteristic function Φ_Y of Y. (Pay attention to the sign of b, i.e. $\Phi_X(u) = \Phi_{-X}(u) = \Phi_X(-u)$ by symmetry.)
    (c) Let $Y_1 \simeq \mathcal{C}(a_1, b_1)$ and $Y_2 \simeq \mathcal{C}(a_2, b_2)$ be independent Cauchy r.v.'s. Show that the sum $Y := Y_1 + Y_2$ is also Cauchy: $Y \simeq \mathcal{C}(a,b)$. Compute a and b from a_1, a_2, b_1 and b_2.

25. With fixed λ, for each integer $n \geq \lambda$, let $X_{1,n}$, $X_{2,n}$,..., $X_{n,n}$ be independent random variables such that

$$P[X_{i,n} = 1] = \frac{\lambda}{n}$$
$$P[X_{i,n} = 0] = 1 - \frac{\lambda}{n}.$$

    Let $Y_n = X_{1,n} + X_{2,n} + ... + X_{n,n}$.
    (a) Find Φ_{Y_n}, the characteristic function of Y_n.
    (b) One can interpret Y_n as the number of successes in n independent Bernoulli trials with success probability λ/n. Verify that Y_n has a binomial distribution! Compute $\mathbb{E}[Y_n]$!
    (c) Find the limit of Φ_{Y_n} as $n \to \infty$. What distribution does it correspond to?
    Hint: use the fact that $(1 + \frac{a}{n})^n$ converges to e^a as n tends to infinity.

[1]This notation is not common and should only be used in this homework set.