ELEC 533 Homework 9

Due date: In class on Wednesday, November 27, 2002

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33. The power spectral density $S_X(\nu)$ of a wss process X_t is defined as the Fourier transform of its auto-correlation function $R_X(\tau)$. Show the following relations:

    (a) If $Y_t = aX_t$ then $R_Y(\tau) = |a|^2 R_X(\tau)$ and $S_Y(\nu) = |a|^2 S_X(\nu)$.

    (b) Let U_t and V_t be orthogonal processes, meaning that $\mathbb{E}[U_t V_s] = 0$ for all t and s. Set $W_t = U_t + V_t$. Show that $R_W(\tau) = R_U(\tau) + R_V(\tau)$ and conclude that $S_W(\nu) = S_U(\tau) + S_V(\tau)$.

    (c) Let X_t be a zero mean process, meaning that $\mu_X(t) = 0$. Set $Z_t = X_t + b$. Show that $R_Z(\tau) = R_X(\tau) + |b|^2$ and $S_Z(\nu) = S_X(\nu) + 2\pi|b|^2\delta(\nu)$.

    (d) If A_t is a wss process such that $R_A(\tau) = -R''_X(\tau)$ then $S_A(\nu) = \nu^2 S_X(\nu)$. (Here, R'' denotes the second derivative.)

    Hint: use known properties of the Fourier transform. The answers can be very short.

34. Compute the spectral density for processes with the following auto-correlation:

    (a) $R_X(\tau) = \max(1 - |\tau|/T, 0)$. HINT: R_X is the convolution of two extremely simple functions with extremely simple Fourier transforms.

35. Let $\{X_t\}$ be a continuous time, but discrete valued Markov process. For simplicity we assume that the values are integers. As in the case with continuous values treated in class, we define the process via its initial distribution $P[X_0 = k]$ at time 0 and the transition probabilities

    $$P[X_t = k | X_s = n]$$

    for any $t > s$. More precisely, the marginals are

    $$P[X_t = k] = \sum_{n=1}^{\infty} P[X_t = k | X_0 = n] \cdot P[X_0 = n]$$

    and the f.d.d. are

    $$P[X_{t_1} = k_1, \ldots X_{t_n} = k_n] = P[X_{t_n} = k_n | X_{t_{n-1}} = k_{n-1}] \cdots P[X_{t_2} = k_2 | X_{t_1} = k_1] \cdot P[X_{t_1} = k_1]$$

    As in the continuous-valued case done in class, this defines a consistent f.d.d. if and only if the Chapman-Kolmogorov equations are satisfied. Here, these Chapman-Kolmogorov equations read as:

    $$\sum_{n=1}^{\infty} P[X_t = k | X_s = n] \cdot P[X_s = n | X_r = m] = P[X_t = k | X_r = m].$$

    (a) Check that the Chapman-Kolmogorov equations imply consistency. To this end, assume that the Chapman-Kolmogorov equations hold and show that for any $j = 1, \ldots n$

    $$\sum_{k_j=1}^{\infty} P[X_{t_1} = k_1, \ldots X_{t_n} = k_n] = P[X_{t_1} = k_1, \ldots, X_{t_{j-1}} = k_{j-1}, X_{t_{j+1}} = k_{j+1}, \ldots, X_{t_n} = k_n].$$

(b) Consider the following particular transition probability where $t > s$ and where k and n are positive integers:

$$P[X_t = k | X_s = n] = \begin{cases} 0 & \text{if } k < n \\ e^{-\lambda(t-s)}(\lambda(t-s))^{k-n}/(k-n)! & \text{if } k \geq n. \end{cases}$$

In other words, given that $X_s = n$ the *increment* $X_t - X_s$ is a Poisson random variable of mean $\lambda(t-s)$.
Verify that the Chapman-Kolmogorov equations are satisfied. Hint 1: recall that

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}.$$

Hint 2: be careful about noting when $P[X_t = k | X_s = n]$ is zero.

(c) With the same transition probabilities as before add the initial distribution $P[X_0 = 0] = 1$. Show that X_t is a Poisson variable of mean λt.
Note: In this case, it is possible to show that X_t is actually a Poisson renewal process. This is not obvious, since one needs to construct arrival times T_k of events which should be spaced by independent waiting times. A first step towards this goal would be to show that the increments $X_t - X_s$ over disjoint time intervals are independent.