ELEC 533 Homework 3

Due date: In class on Tuesday, September 28th

Instructor: Dr. Rudolf Riedi

9. Suppose $X \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. X is a r.v. uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Let $Y = \tan(X)$. The distribution of Y can be computed in two different ways.

   (a) Compute $P[Y \le y]$ from first principles. In other words, show that $F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\arctan(y)$ by direct computation. Then infer $f_Y(y)$ from $F_Y(y)$.

   (b) Use the following formula for the density of $g(X)$ which is valid if g is differentiable and strictly increasing

   $$f_Y(g(x)) = \frac{f_X(x)}{g'(x)}, \qquad \text{or} \qquad f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y).$$

   Note that Y is a Cauchy r.v.

10. (a) Let X be a continuous r.v. Using the definition of expectation and the rules of integration derive

   $$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b,$$

   where a and b are constants. Also show that

   $$\mathbb{E}[(aX + b)^2] = a^2\,\mathbb{E}[X^2] + 2ab\,\mathbb{E}[X] + b^2.$$

   Conclude that $\text{var}(aX + b) = a^2\text{var}(X)$.

   (b) Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = aX + b$. The mean and variance of Y are given by 10a. Find the p.d.f of Y. Comment on the distribution of Y.

   (c) Repeat 10b for $X \sim U(0, 2\pi)$.

11. Suppose that X (signal) is a binary r.v. with $P[X = 1] = \alpha$ and $P[X = -1] = \beta = 1 - \alpha$. Suppose the r.v. N is normal, i.e. $N \sim \mathcal{N}(0, \sigma^2)$ (noise). Assume also that N and X are independent, meaning that the events $\{X = a\}$ and $\{N \le b\}$ are independent for all a and b. We are interested in the r.v. $Y = X + N$ (noisy observation).

   (a) Compute $P[Y \le y | X = 1]$ and $P[Y \le y | X = -1]$.

   (b) Using this and the law of total probability find $F_Y(y)$.

   (c) Compute f_Y from F_Y and relate it to known density functions.

   (d) To infer the signal X from the observation Y the following strategy is used:

   - If $Y \ge \gamma$, we infer that $X = 1$,
   - if $Y < \gamma$, we infer that $X = -1$,

   where γ is some number we will decide on later (next question). We make an error in the inference if $Y \ge \gamma$ and $X = -1$, or if $Y < \gamma$ and $X = 1$. Compute the probability p_e that this happens.

   (e) Find the γ that minimizes the probability of error p_e.

12. Suppose X and Y are joint r.v.'s with

   $$f_{XY}(x, y) = \begin{cases} e^{-y} & x > 0, y > x \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $\iint f_{XY}(x, y)\, dy\, dx = 1$.

   (b) Find f_X and f_Y.

   (c) Are X and Y independent?