ELEC 533 Homework 4

Due date: In class on Tuesday, October 12th

Instructor: Dr. Rudolf Riedi

13. Consider the experiment of tossing three coins independently. The outcomes are then $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega = \{0,1\}^3$, where ω_n is the outcome of the n-th toss, $\omega_n = 1$ for heads, $\omega_n = 0$ for tails. Let $X_n(\omega) = \omega_n$, $Y = X_1 + X_2$ and $Z = X_1 + X_2 + X_3$.

    (a) Let $\sigma(X_1, X_2)$ be the smallest σ-algebra which makes both X_1 and X_2 measurable. Indicate the partitions which define $\sigma(X_1, X_2)$ and $\sigma(Y)$ by drawing a picture in the (ω_1, ω_2)-plane.

    (b) Is $\sigma(X_1, X_2) = \sigma(Y)$?

    (c) Compute $\mathbb{E}[Z|Y]$.

    (d) Compute $\mathbb{E}[Z|X_1]$.

    (e) Show that $\mathbb{E}[\mathbb{E}[Z|Y]|X_1] = \mathbb{E}[Z|X_1]$.

14. Let X, Y, and Z be random variables with joint CDF $F_{XYZ}(x,y,z) = P[X \le x, Y \le y, Z \le z]$. Assume that they have the joint density

$$f_{XYZ}(x,y,z) = \frac{\partial^3}{\partial x \partial y \partial z} F_{XYZ}(x,y,z)$$

meaning that $F_{XYZ}(a,b,c) = \int_{-\infty}^{a} \int_{-\infty}^{b} \int_{-\infty}^{c} f_{XYZ}(x,y,z) dx dy dz$.

   (a) Show that the *joint marginal* density

$$f_{XY}(x,y) = \int_{-\infty}^{\infty} f_{XYZ}(x,y,z) dz$$

   is indeed a valid density function. Similarly, $f_{XZ}(x,z)$ and $f_{YZ}(y,z)$ are valid density functions.

   (b) Show that the marginal density

$$f_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XYZ}(x,y,z) dy dz$$

   is indeed a valid density function.

   (c) For any fixed pair (y,z) with $f_{YZ}(y,z) \neq 0$ show that the *conditional* density

$$f_{X|YZ}(x\,|y,z) = \frac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)}$$

   is indeed a valid density function (x is the variable, y and z are fixed).

   (d) Show the 'law of total probability for densities':

$$\int_{-\infty}^{\infty} f_{X|YZ}(x\,|y,z) \cdot f_{Y|Z}(y\,|z) dy = \frac{f_{XZ}(x,z)}{f_Z(z)} \stackrel{\text{def}}{=} f_{X|Z}(x\,|z).$$

   (e) We define

$$\mathbb{E}[X|Y,Z] := \int_{-\infty}^{\infty} x f_{X|YZ}(x) dx.$$

   Conclude from (14d) that

$$\mathbb{E}[X|Z] = \mathbb{E}[\mathbb{E}[X|Y,Z]|Z]$$

15. In class we have seen that $\mathbb{E}[\mathbb{E}[U|V]] = \mathbb{E}[U]$.

(a) Conclude that

$$\mathbb{E}[(U - \mathbb{E}[U|V])^2] = \mathrm{Var}(U - \mathbb{E}[U|V]).$$

For the remainder recall the random variables of Problem 13.

(b) Show that $\mathbb{E}[Z|Y] = \mathbb{E}[Z|X_1, X_2]$.

(c) Without actually computing their values show that

$$\mathrm{Var}(Z - \mathbb{E}[Z|X_1]) = 2\mathrm{Var}(Z - \mathbb{E}[Z|X_1, X_2]).$$

This illustrates that the variance of error grows as we take a guess at Z with less knowledge (Note that $\sigma(X_1) \subset \sigma(X_1, X_2)$).

(d) Is (15c) still true if we throw dice instead of coins?

16. Let the joint density of the pair of random variables (X, Y) be given by

$$f_{XY}(x, y) = \begin{cases} y \exp(-xy) & \text{if } x > 1 \text{ and } y > 0, \\ 0 & \text{else.} \end{cases}$$

(a) Compute the marginal densities f_X and f_Y.

(b) Are X and Y independent? Show your reasoning.

(c) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.