# ELEC 533 Homework 5

Due date: In class on Thursday, October 21th

Instructor: Dr. Rudolf Riedi

17. (a) Let $X \simeq \text{Poiss}(\lambda)$ (see Homework problem 7 (a)). Compute the characteristic function $\Phi_X$ of $X$.
    (b) Let $X_1 \simeq \text{Poiss}(\lambda_1)$ and $X_2 \simeq \text{Poiss}(\lambda_2)$ be independent r.v.'s. Show that $X_1+X_2 \simeq \text{Poiss}(\lambda_1+\lambda_2)$. Thus, the sum of two independent Poisson r.v.'s is also Poisson. (HINT: Use the characteristic function.)
    (c) Let $X \simeq \exp(\lambda)$ (see Homework problem 7 (c)). Compute the characteristic function $\Phi_X$ of $X$.
    (d) Is the sum of two independent exponential r.v.'s an exponential r.v.?

18. (a) Let $X \simeq \mathcal{C}(0,1)$:[1] a standard Cauchy r.v. which is given through the density

$$f_X(x) = \frac{1}{\pi(1+x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

    Verify that the characteristic function $\Phi_X(u)$ is given by

$$\Phi_X(u) = e^{-|u|}.$$

    (b) Let $Y \simeq \mathcal{C}(a,b)$ be a general Cauchy r.v., meaning that $Y = a+bX$, where $X \simeq \mathcal{C}(0,1)$ and where $a$ and $b$ are constants. Compute the characteristic function $\Phi_Y$ of $Y$. (Pay attention to the sign of $b$, i.e. $\Phi_X(u) = \Phi_{-X}(u) = \Phi_X(-u)$ by symmetry.)
    (c) Let $Y_1 \simeq \mathcal{C}(a_1,b_1)$ and $Y_2 \simeq \mathcal{C}(a_2,b_2)$ be independent Cauchy r.v.'s. Show that the sum $Y := Y_1 + Y_2$ is also Cauchy: $Y \simeq \mathcal{C}(a,b)$. Compute $a$ and $b$ from $a_1$, $a_2$, $b_1$ and $b_2$.
    (d) Recall the definition of a (symmetrical) $\alpha$-stable r.v. $(0 < \alpha \leq 2)$: $X \simeq S\alpha S(\mu,\sigma)$ if

$$\Phi_X(u) = e^{(i\mu u - |\sigma|^\alpha |u|^\alpha)},$$

    where $\mu$ is the position parameter, and $\sigma$ is the scale parameter. Proceeding as in (18c) show that the sum of two independent $S\alpha S$ r.v.'s is also $S\alpha S$. (Note that a Cauchy r.v. is $S\alpha S$ with $\alpha = 1$.)
    (e) Let $X \simeq S\alpha S(\mu,\sigma)$ and assume that $\alpha > 1$. Show that $\mu = \mathbb{E}[X]$. (Note: When $\alpha \leq 1$, then $\mathbb{E}[|X|] = \infty$. Hence $\mathbb{E}[X]$ is not defined.)

19. Let $X$ and $Y$ be standard jointly Gaussian r.v.'s ($\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$) with joint density

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right),$$

    where $\rho$ is a constant.
    (a) Compute the correlation coefficient $\text{cov}(X,Y)/\sqrt{\text{Var}(X)\text{Var}(Y)}$.
    (b) Conclude that if two jointly Gaussian r.v.'s are uncorrelated, then they are independent.

20. Let $X \simeq \mathcal{N}(\mu,\sigma^2)$ be a Gaussian r.v. and set $Y = e^X$. $Y$ is called log-normal.
    (a) Compute $\mathbb{E}[Y]$. Use this formula to compute $\text{Var}(Y)$.
    (b) Verify that $\mathbb{E}[e^X] > e^{\mathbb{E}[X]}$.

[1] This notation is not common and should only be used in this homework set.