ELEC 533 Homework 6

Due date: In class on Thursday, October 28th

Instructor: Dr. Rudolf Riedi

21. Let $\Omega = [0, 1]$ be sample space (universal set), where the event class (σ-algebra) are all Borel subsets of Ω. The probability distribution on Ω is uniform. Define a sequence of random variables X_m from Ω to $\mathbb{R}$ as follows: Write m uniquely in the form $m = 2^n + k$ with $0 \le k \le 2^n - 1$ and set

$$X_m = X_{2^n+k}(\omega) = \begin{cases} 1 & \text{for } \omega \text{ in } [k2^{-n}, (k+1)2^{-n}], \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that X_m converges in probability but not almost surely.

   (b) Find a subsequence X_{m_l} so that
$$X_{m_l} \xrightarrow{\text{a.s.}} 0$$

   (c) Find a random variable Y so that
$$|X_m| \le Y,$$
and
$$E[|Y|^2] < \infty$$

   (d) A theorem states that a sequence which converges in probability (see 21a) and is mean square dominated (see 21c) does converge in the mean square sense, i.e.
$$X_m \xrightarrow{\text{m.s.}} 0.$$
Verify this for the given sequence X_m by showing
$$E[|X_m|^2] \longrightarrow 0$$

   (e) Define
$$Y_{2^n+k} = 2^n X_{2^n+k}$$
Show that
$$Y_m \xrightarrow{\text{i.p.}} 0.$$
Also show that Y_m does not converge in the m.s..

22. With fixed λ, for each integer $n \ge \lambda$, let $X_{1,n}$, $X_{2,n}$,..., $X_{n,n}$ be independent random variables such that
$$P[X_{i,n} = 1] = \frac{\lambda}{n}$$
$$P[X_{i,n} = 0] = 1 - \frac{\lambda}{n}.$$
Let $Y_n = X_{1,n} + X_{2,n} + ... + X_{n,n}$.

   (a) Find Φ_{Y_n}, the characteristic function of Y_n.

   (b) Find the limit of Φ_{Y_n} as $n \longrightarrow \infty$. What distribution does it correspond to?