ELEC 533 Homework 7

Due date: In class on Tuesday, November 2nd

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23. There is a theorem which states that if there is a random variable Y with $E[|Y|^2] < \infty$ such that $|X_m| \leq Y$, almost surely, and if X_m converges in probability then X_m converges also in the mean square sense. Your task is to prove a simplified version of this fact which goes as follows:
Suppose that
$$X_m \xrightarrow{\text{i.p.}} X$$
and suppose that there is a constant c such that $|X_m| \leq c$ for all n, then
$$X_m \xrightarrow{\text{m.s.}} X.$$

24. Suppose that
$$X_m \xrightarrow{\text{D}} X$$
and that there is a constant c such that $P[X = c] = 1$. Show that
$$X_m \xrightarrow{\text{i.p.}} X$$

25. This problem looks long but is easy. Consider the experiment of an infinite series of independent Bernoulli trials with success probability p and let
$$X_n = \begin{cases} 1 & \text{if the } n\text{-th experiment is successful} \\ 0 & \text{if the } n\text{-th experiment fails.} \end{cases}$$
As explained in class, this means that the universal set is the set of all 0-1-sequences: $\Omega = \{0,1\}^{\mathbb{N}}$ and the probability on Ω is uniquely defined by the probabilities
$$P[\{\omega = (\omega_1, \omega_2, \ldots) : \omega_n = \text{success}\}] = P[X_n = 1] = p$$
for all fixed n. Define $S_m := \sum_{n=1}^{m} X_n \cdot 2^{-n}$.

    (a) Show that S_m converges almost surely. Let us denote its limit by T.

    (b) Show that the smallest possible value of T is 0, and the largest is 1.

    (c) Show that S_m converges in the mean square sense (use (25b) or proceed directly).

    (d) Let us compute F_T. Conclude from (25b) that $F_T(a) = 0$ for $a \leq 0$ and $F_T(a) = 1$ for $a \geq 1$. Further, let k be an integer from $\{0, 1, \ldots, 2^m - 1\}$. Then, $k2^{-m}$ is a rational number between 0 and 1 with binary expansion of length m, say $k2^{-m} = (b_1 b_2 \ldots b_m)_2 = \sum_{n=1}^{m} b_n \cdot 2^{-n}$. Denote the number of 1-s in this expansion by $\phi(k)$. Show that
    $$P[k2^{-m} \leq T < (k+1)2^{-m}] = P[X_1 = b_1, X_2 = b_2, \ldots, X_m = b_m] = p^{\phi(k)}(1-p)^{m-\phi(k)}.$$

    (e) Suppose that $p \neq 0$ and $p \neq 1$. Show that T is a continuous r.v. Hint, use (25d) to show that
    $$P[k2^{-m} \leq T < (k+1)2^{-m}] \leq (\max(p, 1-p))^m .$$
    Note that as a consequence of continuity (25d) determines $F_T(a)$ for all a.

    (f) Describe T and F_T in the cases $p = 0$ and $p = 1$.

    (g) For the special case $p = 1/2$ show that T is uniformly distributed in the interval $[0, 1]$. Because F_T is continuous, it is enough to show that $F_T(a) = a$ for all dyadic numbers $a \in [0, 1]$, i.e. for all $0 \leq a = k2^{-n} \leq 1$.

Remark: When $p \notin \{0, 1/2, 1\}$, then F_T is continuous but has no density. It is called a singular distribution and is the simplest case of a multifractal.