ELEC 533 Homework 8

Due date: In class on Tuesday, November 9th

Instructor: Dr. Rudolf Riedi

26. Let X_n be a sequence of independent, identically distributed (iid) symmetrical stable variables, more precisely $X_n \simeq S\alpha S(\mu, \sigma)$ $(0 < \alpha \leq 2)$, which means that

$$\Phi_{X_n}(u) = e^{(i\mu u - |\sigma|^\alpha |u|^\alpha)},$$

where α, μ and σ are fixed paramters (recall homework 18).

   (a) Use homework (18 c) to show that

$$Z_n := \frac{X_1 + \ldots + X_n - n\mu}{n^{1/\alpha}}$$

   are distributed as $X_n - \mu$, i.e. $Z_n \simeq S\alpha S(0, \sigma)$.

   (b) Recall that Cauchy variables are the special case of $S\alpha S$ variables with $\alpha = 1$. Use (26a) to show that the Law of Large Numbers (LLN) does not hold for a sequence of iid Cauchy random variables. Which assumption(s) of the LLN stated in class are violated?

   (c) Still using (26a) show that the Central Limit Theorem does not hold for a sequence X_n of iid $S\alpha S$ random variables ($\alpha < 2$). Which assumption(s) of the CLT stated in class are violated?

   (d) Still for the sequence X_n of (iid) $S\alpha S$ random variables: Guess for which values of the parameter α does the weak LLN hold? Give a heuristic answer only. No reasoning required.

27. Let $I(\omega)$ be a discrete random variable:

$$\sum_{n=1}^{\infty} P[I = a_n] = 1.$$

Define the random process $\{X_t(\omega)\}_{t \in T}$ by

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = I(\omega) \\ 0 & \text{else.} \end{cases}$$

   (a) Compute the finite dimensional distributions (fdd) of $\{X_t(\omega)\}_{t \in T}$.

   (b) Is $\{X_t(\omega)\}_{t \in T}$ uniquely determined by the fdd? More precisely, if $Y_t(\omega)\}_{t \in T}$ is another process with exactly the same fdd, is it true that the paths of X_t and Y_t occur with the same probabilities?

28. Let X_n be a sequence of iid random variables.

   (a) Show that X_n is not a martingale unless the sequence is constant almost surely. HINT: Start by assuming that X_n *is* a martingale and show that then $X_n = \mathbb{E}[X_{n+1}] = \mathbb{E}[X_1]$.

   (b) Assume that $\mathbb{E}[X_n] = 0$ and let $Y_n = a_1 X_1 + \ldots a_n X_n$ where $a_1, a_2, \ldots$ are fixed numbers. Show that Y_n is a martingale. If we want Y_n to be bounded in $\mathcal{L}^2$, i.e. if we want that there is a constant $M < \infty$ such that $\mathbb{E}[|Y_n|^2] (= \mathrm{Var}(Y_n)) \leq M$, what condition on $\mathrm{Var}(X_n)$ and the sequence a_n do we have to impose?

   (c) Assume that $\mathbb{E}[X_n] = 1$ and let $Z_n = X_1 \cdots X_n$. Show that Z_n is a martingale. Assume in addition that $P[X_n \geq 0] = 1$. Show that Z_n bounded in $\mathcal{L}^1$, i.e. there a constant $M < \infty$ such that $\mathbb{E}[|Z_n|] \leq M$, but that it is not bounded in $\mathcal{L}^2$ (unless $\mathrm{Var}(X_n) = 0$).

Take home lesson: think of martingales not as a sequence of independent random variables but as a sequence of random variables with 'fairly independent' (orthogonal) *innovations* or increments. A real world example of Y_n is given by a fair game where the X_n are the amount of money a player wins/looses (depending on the sign) and where Y_n is his total win/loss after n games.

29. The term martingale originates from gambling, where it meant a particular stragedy of putting bets in roulette according to Y_n as described in (29a) below. Here, we show that the 'martingale' stragedy does not give an advantage. Moreover, an even more subtle 'shadowing stragedy of a martingale' (see Z_n below) does not improve chances of winning.

(a) Let Y_n be the capital of gambler A after n games which work as follows: A will bet the amount of a_n on HEAD (or on TAIL), then a fair coin is tossed, and his capital will change by $X_n = \pm a_n$ depending on whether the face he has bet on appears or does not appear. The gambler A has adopted the following 'martingale bet' stragedy: He will bet $a_1 = 1$ in the first game. If he looses, he will double to $a_2 = 2$, in the next game to $a_3 = 4$ and so forth until he wins. Once he won, he will start over with bet $a_k = 1$, doubling it until he wins. In short, $Y_n = Y_0 + X_1 + \ldots + X_n$ where Y_0 is the starting capital. Note that the X_n are not independent: Y_n hopes to win at least the amount of 1 if he can 'wait it out'. Show that Y_n is a martingale.
It follows that $\mathbb{E}[Y_n] = Y_0$ for all n: the stragedy does not improve his chances to win.

(b) Assume that gambler A has a stragedy of betting that results in a capital Y_n which is a martingale not neccessarily the one above). Let Z_n be the capital of a second gambler B who observes gambler A and tries to make a winning out of it. His betting stragedy is as follows: He sets himself two limits g and s with $g < s$. He bets the amount $b_n = 0$ until Y_n drops for the first time under g. Then he bets exactly the same way as gambler A does, until Y_n reaches the level s the first time. Now, repeats the procedure, putting bets $b_n = 0$ until Y_n drops again below g, and betting as A does until Y_n reaches the level s again, etc. In this way, gambler B plans to join gambler A only in his winning periods, hoping to collect the amount of at least $s - g$ in each such period and coming out with an almost sure win. Show that $\mathbb{E}[Z_{n+1}|Y_1, \ldots Y_n] = Z_n$, thus $\mathbb{E}[Z_n] = Z_0$. HINT: Note that $Z_{n+1} - Z_n$ is either 0 or equal to $Y_{n+1} - Y_n$ and conclude that $\mathbb{E}[Z_{n+1} - Z_n|Y_1, \ldots Y_n] = 0$.

Note: the observation that $\mathbb{E}[Z_n] = Z_0$ lies at the heart of a proof of the martingale convergence theorem (i): Speaking very loosely it allows to conclude that Y_n cannot have 'too many' up-crossings from g to s, thus, it cannot 'vary too much at infinity', thus, it converges.