STAT 650 Homework 2

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Due date: Tuesday, Feb. 21, 2006

Bring to class or hand in to Dr. Riedi, Duncan Hall 2082

4. A sequence of random variables $\{U_n\}_n$ is called a *martingale* with respect to $\mathcal{F}_n$ iff for each n,

   (a) $\mathcal{F}_n$ is a subfield of $\mathcal{F}$ and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$,

   (b) U_n is in L^1 and $\mathcal{F}_n$-measurable, and

   (c) $\mathbb{E}[U_{n+1}|\mathcal{F}_n] = U_n$.

   Let X_n be iid zero mean random variables, and let Y_n be iid mean 1 random variables. Let $S_n = X_1 + \ldots + X_n$, let $T_n = Y_1 \ldots Y_n$ and assume that S_n and T_n are in L^1 for all n.

   (a) Assume that U_n is a martingale. Show that $\mathbb{E}[U_{n+1}] = \mathbb{E}[U_n] = \mathbb{E}[U_1]$. Hint: use property (c) of a martingale.

   (b) Show, that S_n is a martingale with respect to $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$.

   (c) What is wrong about the following statement (consult the properties (a)-(c) of a martingale): "S_n is a martingale with respect to $\mathcal{H}_n := \sigma(S_n)$."

   (d) Show that T_n is a martingale with respect to $\mathcal{G}_n := \sigma(Y_1, \ldots, Y_n)$.

5. Check whether the following processes X_t are martingales w.r.t. the indicated filtration.

   (a) $X_t = W_t + 4t$ w.r.t. $\mathcal{W}_t = \sigma(W_s : s \le t)$.
   Hint: consider $\mathbb{E}[X_t]$.

   (b) $X_t = W_t \tilde{W}_t$ w.r.t. $\mathcal{G}_t = \sigma(W_s, \tilde{W}_s : s \le t)$, where W_t and $\tilde{W}_t$ are two independent BM.

6. Show that the following processes X_t are martingales w.r.t. $\mathcal{W}_t = \sigma(W_s : s \le t)$.

   (a) $X_t = t^2 W_t - 2\int_0^t s W_s ds$.
   Hint: Use Ito's formula to identify X_t as an Ito integral.

   (b) $X_t = W_t^3 - 3tW_t$.
   Hint: proceed as before, but in two steps

7. Let X_t and Y_t be Ito processes on $\mathbb{R}$ (one dimensional, w.r.t. the same BM). Establish the general form of "integration by parts":

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

   which means that

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s$$

   Hint: Use Ito's formula.

8. Solve the Ornstein-Uhlenbeck differential equation (or Langevin equation):

$$dX_t = \mu X_t dt + \sigma dW_t$$

   Hint: Apply Ito's formula to $g(t, x) = e^{-t\mu} x$; here, $e^{-t\mu}$ is called an integrating factor.