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Multifractal Processes

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ABSTRACT

This paper has two main objectives. First, it develops the multifractal formalism in a context suitable for both, measures and functions, deterministic as well as random, thereby emphasizing an intuitive approach. Second, it carefully discusses several examples, such as the binomial cascades and self-similar processes with a special eye on the use of wavelets. Particular attention is given to a novel class of multifractal processes which combine the attractive features of cascades and self-similar processes. Statistical properties of estimators as well as modelling issues are addressed.

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1 Introduction and Summary

Fractal processes have been instrumental in a variety of fields ranging from the theory of fully developed turbulence [73, 64, 36, 12, 7], to stock market modelling [28, 68, 69, 80], image processing [61, 21, 104], medical data [2, 98, 11] and geophysics [36, 65, 47, 92]. In networking, models using fractional Brownian motion (fBm) have helped advance the field through their ability to assess the impact of fractal features such as statistical self-similarity and long-range dependence (LRD) to performance [60, 81, 90, 89, 96, 34, 88].

Roughly speaking, a fractal entity is characterized by the inherent, ubiquitous occurrence of irregularities which governs its shape and complexity. The most prominent example is certainly fBm $B_H(t)$ [71]. Its paths are almost surely continuous but not differentiable. Indeed, the oscillation of fBm in any interval of size δ is of the order δ^H where $H \in (0,1)$ is the self-similarity parameter:

$$B_H(at) \stackrel{fd}{=} a^H B_H(t). \tag{1.1}$$

Reasons for the success of fBm as a model of LRD may be seen in the simplicity of its scaling properties which makes it amendable to analysis. The fact of being Gaussian bears further advantages. However, the scaling law (1.1) implies also that the oscillations of fBm at fine scales are uniform* which comes as a disadvantage in various situations (see Figure 1). Real world signals often possess an erratically changing oscillatory behavior (see Figure 2) which have earned them the name *multifractals*, but which also limits the appropriateness of fBm as a model. This rich structure at fine scales may serve as a valuable indicator, and ignoring it might mean to miss out on relevant information (see references above).

This paper has two objects. First, we present the framework for describing and detecting such a multifractal scaling structure. Doing so we survey local and global multifractal analysis and relate them via the multifractal formalism in a stochastic setting. Thereby, the importance of higher order statistics will become evident. It might be especially appealing to the reader to see wavelets put to efficient use. We focus mainly on the analytical computation of the so-called multifractal spectra and on their mutual relations. Thereby, we emphasize issues of observability by striving for formulae which hold for all or almost all paths and by pointing out the necessity of oversampling needed to capture certain rare events. Statistical properties of estimators of multifractal quantities as well as modelling issues are addressed elsewhere (see [41, 3, 40] and [68, 89, 88]).

Second, we carefully discuss basic examples as well as *Brownian motion in multifractal time*, $B_{1/2}(\mathcal{M}(t))$. This process has recently been suggested as a model for stock market exchange by Mandelbrot who argues that oscillations in exchange rates occur in multifractal 'trading time' [68, 69]. With the theory developed in this paper, it becomes an easy task to explore $B_{1/2}(\mathcal{M}(t))$ from the multifractal point of view, and with

*This property is also known as the Lv́ey modulus of continuity in the case of Brownian motion. For fBm see [5, Thm. 8.3.1.].

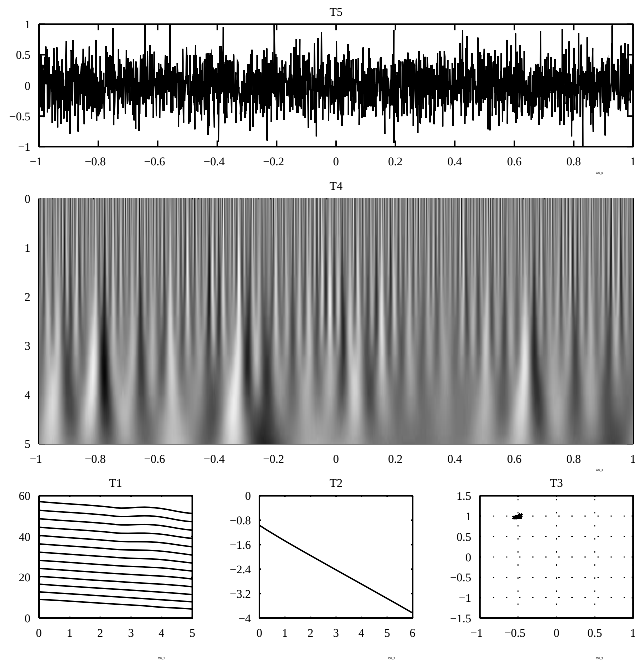

FIGURE 1. *Fractional Brownian motion, as well as its increment process called fGn (displayed on top in T5), has only one singularity exponent* $h(t) = H$, *a fact which is represented in the linear partition function* τ *(see T2) and a multifractal spectrum (see T3) which consists of only one point: for fBm* $(H, 1)$ *and for fGn* $(H - 1, 1)$. *For further details on the plots see (1.9), (1.6) and Figure 7.*

little more effort also more general multifractal 'subordinators'. The reader interested in these multifractal processes may wish, at least at first reading, to content himself with the notation introduced on the following few pages, skip the sections which deal more carefully with the tools of multifractal analysis, and proceed directly to the last sections. The remainder of this introduction provides a summary of the contents of the paper, following roughly its structure.

1.1 Singularity Exponents

In this work, we are mainly interested in the geometry or local scaling properties of the paths of a process $Y(t)$. Therefore, all notions and results concerning paths will apply to functions as well. The study of fine scale properties of functions (as opposed to measures) has been pioneered in the work of Arneodo, Bacry and Muzy [7, 78, 79, 1, 2, 80], who were also the first to introduce wavelet techniques in this context. For simplicity of the presentation we take $t \in [0, 1]$. Extensions to the real line $\mathbb{R}$ as well as to higher dimensions, being straightforward in most cases, are indicated.

A typical feature of a fractal process $Y(t)$ is that it has a non-integer degree of differentiability, giving rise to an interesting analysis of its local *Hölder exponent* $H(t)$

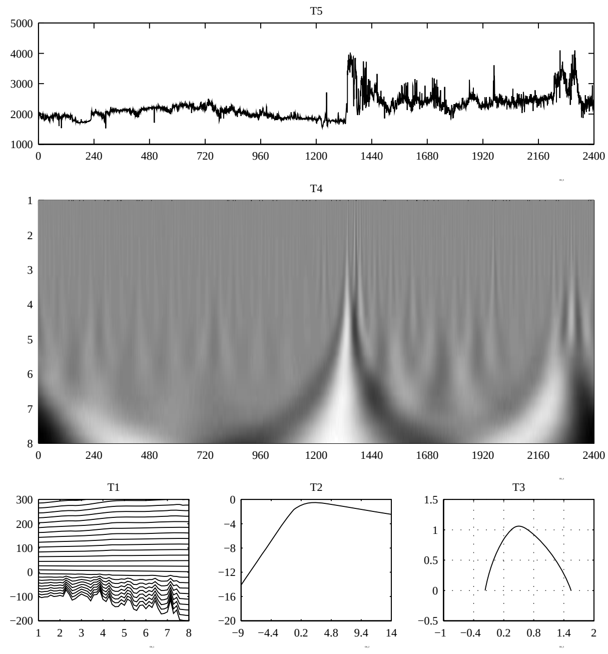

FIGURE 2. *Real world signals such as this geophysical data often exhibit erratic behavior and their appearance may make stationarity questionable. One such feature are 'trends' which sometimes can be explained by strong correlations (LRD). Another such feature are the sudden jumps or 'bursts' which in turn are a typical for multifractals. For such signals the singularity exponent $h(t)$ depends erratically on time t, a fact which is reflected in the concave partition function τ (see T2) and a multifractal spectrum (see T3) which extends over a non-trivial range of singularity exponents.*

which is roughly defined through

$$|Y(t') - P(t')| \simeq |t' - t|^{H(t)} \tag{1.2}$$

for some polynomial P which in nice cases is simply the Taylor polynomial of Y at t. A rigorous definition is given in (2.1).

Provided the polynomial is *constant*, $H(t)$ can be obtained from the limiting behavior of the so-called *coarse Hölder exponents*, i.e.,

$$h_\varepsilon(t) = \frac{1}{\log \varepsilon} \log \sup_{|t'-t|<\varepsilon} |Y(t') - Y(t)|. \tag{1.3}$$

For rigorous statements we refer to (2.2) and lemma 2.3.

However, as the example $t^2 + t^{2.7}$ shows, the use of $h_\varepsilon(t)$ is ineffective in the presence of polynomial trends. Then, $h_\varepsilon(t)$ will reflect the lowest non-constant term of the Taylor polynomial of Y at t. For this reason, and also to avoid complications introduced through the computation of the supremum in (1.3), one may choose to employ *wavelet* *decompositions* or other tools of time frequency analysis. Properly chosen wavelets are blind to polynomials and due to their scaling properties they contain information on the Hölder regularity of Y [51, 23]. Their application in multifractal estimation has been pioneered by [7, 53, 30]. Furthermore, wavelets provide unconditional basis for several regularity spaces such as Besov spaces (see (2.14) and (6.2)) whence their use bears further advantages.

Yet, the 'classical' choice of a singularity exponent is

$$\alpha_k^{(n)} = \frac{1}{-n \log 2} \log \big(\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n}) \big). \tag{1.4}$$

It is attractive due to its simplicity and becomes actually quite powerful when studying monotonously increasing processes $\mathcal{M}(t)$, in particular the distribution functions of *singular measures*, such as cascades.

In this chapter we will introduce the exponents $w_k^{(n)}$ emerging from a wavelet based analysis and elaborate on the relation between these different singularity exponents $h_k^{(n)}$, $\alpha_k^{(n)}$ and $w_k^{(n)}$.

1.2 Multifractal Spectra

As indicated we are mainly interested in the geometry or local regularity of the paths of $Y(t)$. Let us fix such a realization for the time-being.

Local analysis

Ideally, one would like to quantify in geometrical as well as statistical sense which values $H(t)$ appear on a given path of the process Y, and how often one will encounter them. Towards the first description one studies the sets

$$E_h^{[a]} = \{t \,:\, H(t) = a\} \tag{1.5}$$

for varying a. Similarly, one could consider sets $E_\alpha^{[a]}$ and $E_w^{[a]}$ defined through the limiting behavior of the singularity exponents $\alpha_k^{(n)}$ or $w_k^{(n)}$, respectively. If no confusion regarding the choice of $h_k^{(n)}$, $w_k^{(n)}$ or $\alpha_k^{(n)}$ can arise, we simply drop the index.

The sets $E^{[a]}$ form a decomposition of the support of Y according to its singularity exponents. We say that Y has a *rich multifractal structure* if these sets $E^{[a]}$ are highly interwoven, each lying dense on the line. Typically, only one of the $E^{[a]}$ has full Lebesgue measure, while the others form dusts, more precisely, sets with non-integer Hausdorff dimension $\dim(E^{[a]})$ [32]. Dimensions are always positive, and the smaller the dimension of a set the 'thinner' the set. In this sense, the function

$$a \mapsto \dim(E^{[a]}) \tag{1.6}$$

gives a compact representation of the complex singularity structure of Y. It has been termed the *multifractal spectrum* of Y and is studied extensively in the 'classical' literature.

To develop some intuition let us consider a differentiable path. To avoid trivialities let us assume that this path and its derivative have no zeros. Then, $\dim(E^{[a]})$-spectrum reduces to the point $(1,1)$. On the other hand, if $H(t)$ is continuous and not constant on intervals then each $E^{[a]}$ is finite and $\dim(E^{[a]}) = 0$ for all a in the range of $H(t)$. A spectrum $\dim(E^{[a]})$ with non-degenerate form is, thus, indeed indication for rich singularity behavior. By this we mean that $H(t)$ changes erratically with t and takes each value a on a rather large set $E^{[a]}$.

Global analysis

A simpler way of capturing the complex structure of a signal is obtained when adapting the concept of box-dimension to the multifractal context. As the name indicates, one aims at an estimate of $\dim(E^{[a]})$ by counting the intervals – or boxes – over which Y increases roughly with the 'right' Hölder exponent. Therefore, we need to introduce grain exponents, a discrete approximation to $h_\varepsilon(t)$ (see (1.3)):

$$h_k^{(n)} := -(1/n)\log_2 \sup\{|Y(s) - Y(t)| \,:\, (k-1)2^{-n} \le s \le t \le (k+2)2^{-n}\} \qquad (1.7)$$

and define the *grain (multifractal) spectrum* as [73, 46, 45, 91]

$$f(a) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log N^{(n)}(a,\varepsilon)}{n \log 2}, \qquad (1.8)$$

where $N^{(n)}(a,\varepsilon) = \#\{k \,:\, |h_k^{(n)} - a| < \varepsilon\}$ counts, how many of the grain exponents $h_k^{(n)}$ are approximately equal to a. Similarly, one may define such spectra for the singularity exponents $\alpha_k^{(n)}$ and $w_k^{(n)}$. If confusion may arise, we will indicate the chosen exponent by writing explicitly $f_h(a)$, $f_\alpha(a)$, or $f_w(a)$.

This multifractal spectrum can be interpreted (at least) in three ways. First, as mentioned already it is related to the notion of *dimensions*. Indeed, a simple argument shows that $\dim(E^{[a]}) \le f(a)$ [94]. The essential ingredient for a proof is the fact that the calculation of $\dim(E^{[a]})$ involves finding an optimal covering of $E^{[a]}$ while $f(a)$ considers only uniform covers. In short, $f(a)$ provides an upper bound on the dimension and, thus, the 'size' of the sets $E^{[a]}$.

Second, (1.8) suggests that the *re-normalized histograms* $(1/n)\log_2 N^{(n)}(a,\varepsilon)$ should all be roughly equal at small scales 2^{-n} to the *scale independent* $f(a)$. It should be remembered that this is foremost (by definition) a property of the *paths* of the given process. We stress this point because it is tempting to argue that –at least under suitable ergodicity assumptions– one should see the marginal distribution of $h_k^{(n)}$ reflected in f. However, one should not overlook that the logarithmic re-normalization implemented in $f(a)$ is aimed at detecting exponential scaling properties rather than the marginals on multiple scales themselves. For fBm (see (1.1)) this re-normalization indeed causes all details of the Normal multi-scale marginals to be washed out into a virtually structureless $f(a)$ which gives notice of the presence of only *one* scaling law, the self-similarity (1.1) with parameter H. Thus, f expresses that fBm is 'mono-fractal', as mentioned above. To the contrary with 'multi-fractal' processes such as multiplicative cascades, for which f reflects the presence of an entire range of scaling exponents (see (5.32)).

The third natural context for the coarse spectrum f is that of Large Deviation Principles (LDP) [29, 91]. Indeed, $N^{(n)}(a,\varepsilon)/2^n$ defines a probability distribution† on $\{h_k^{(n)} : k = 0, \ldots, 2^n - 1\}$. Alluding to the Law of Large Numbers (LLN) we may expect this distribution to be concentrated more and more around the 'most typical' or 'expected' value as n increases. The spectrum $f(a)$ measures how fast the chance $N^{(n)}(a,\varepsilon)/2^n$ to observe a 'deviant' value a decreases, i.e., $N^{(n)}(a,\varepsilon)/2^n \simeq 2^{f(a)-1}$.

The close connection to LDP leads one to study the scaling of 'sample moments' through the so-called *partition function* [45, 46, 36, 91]

$$\tau_h(q) := \liminf_{n\to\infty} \frac{\log S_h^{(n)}(q)}{-n\log 2} \qquad \text{where} \quad S_h^{(n)}(q) := \sum_{k=0}^{2^n-1} 2^{-nqh_k^{(n)}}, \tag{1.9}$$

which are defined for all $q \in \mathbb{R}$. Similarly, replacing $h_k^{(n)}$ by $\alpha_k^{(n)}$, one defines $\tau_\alpha(q)$ and $S_\alpha^{(n)}(q)$. The latter takes on the well-known form of a partition sum

$$S_\alpha^{(n)}(q) = 2^{-nq\alpha_k^{(n)}} = \sum_{k=0}^{2^n-1} \left|Y\left((k+1)2^{-n}\right) - Y(k2^{-n})\right|^q. \tag{1.10}$$

Again similarly, one defines $\tau_w(q)$ and $S_w^{(n)}(q)$ by replacing $h_k^{(n)}$ by wavelet based exponents $w_k^{(n)}$ (see (2.11)). Again, if no confusion on the choice of $h_k^{(n)}$, $w_k^{(n)}$ or $\alpha_k^{(n)}$ can arise, we simply drop the index h, α or w .

1.3 Multifractal Formalism

The theory of LDP suggests $f(a)$ and $\tau(q)$ are strongly related since $2^{-n}S^{(n)}(q)$ is the moment generating function of the random variable $A_n(k) := -nh_k^{(n)}\ln(2)$ (recall footnote †). For a motivation of a formula connection $f(a)$ and $\tau(q)$ consider the heuristics

$$S^{(n)}(q) \;=\; \sum_a \sum_{h_k^{(n)}\simeq a} 2^{-nqh_k^{(n)}} \simeq \sum_a 2^{nf(a)} 2^{-nqa} = \sum_a 2^{-n(qa-f(a))} \simeq 2^{-n\inf_a(qa-f(a))}.$$

Assuming that $\sum_a$ has only finite many terms the last step simply replaces the sum by its strongest term. Making this entire argument rigorous we prove in this paper that

$$\tau(q) = f^*(a) := \inf_a(qa - f(a)). \tag{1.11}$$

Here $(\cdot)^*$ denotes the Legendre transform which is omnipresent in the theory of LDP. Indeed, by applying a theorem due to Gärtner and Ellis [27] and imposing some regularity on $\tau(q)$ theorem 3.5 shows that the family of probability densities defined by $N^{(n)}(a,\varepsilon)/2^n$ satisfies the *full LDP [26] with rate function* f meaning that f is actually a double-limit and $f(a) = \tau^*(a)$. Corollary 4.5 establishes that always

$$f(a) = \tau^*(a) = qa - \tau(q) \qquad \text{at points } a = \tau'(q). \tag{1.12}$$

†Recall that we fixed a path of Y. Randomness is here understood in choosing k.

Going through some of the explicitly calculated examples in Section 5.5 will help demystify the Legendre transform. A tutorial on the Legendre transform in contained in Appendix A of [89].

From (1.11) follows, that $f(a) \leq f^{**}(a) = \tau^*(a)$ and also that $\tau(q)$ is a concave function, hence continuous and almost everywhere differentiable.

1.4 Deterministic Envelope

So far, all that has been said applies to any given function or path of a process. In the random case, one would often like to use a simple analytical approach in order to gain intuition or an estimate of f for a typical path of Y.

To this end we formulate a LDP for the sequence of distributions of $\{h_k^{(n)}\}$ where randomness enters now through choosing $k \in \{0, \ldots, 2^n - 1\}$ as well as through the randomness of the process itself, i.e., through $Y_t(\omega)$ where ω lies in the probability space (Ω, P_Ω). The moment generating function of $A_n(k,\omega) = -nh_k^{(n)}(\omega)\ln(2)$ with k and ω random is $2^{-n}\mathbb{E}_\Omega[S^{(n)}(q)]$. This leads to defining the 'deterministic envelope':

$$T(q) := \liminf_{n\to\infty} \frac{-1}{n} \log_2 \mathbb{E}_\Omega S^{(n)}(q) \tag{1.13}$$

and the corresponding 'rate function' F (see (3.23)). As with the pathwise $f(a)$ and $\tau(q)$ we have here again $T(q) = F^*(q)$. More importantly, it is easy to show that $\tau(q,\omega) \geq T(q)$ almost surely (see lemma 3.9). Thus:

Corollary 1.1. *With probability one the multifractal spectra are ordered as follows: for all a*

$$\dim(E^{[a]}) \leq f(a) \leq \tau^*(a) \leq T^*(a), \tag{1.14}$$

provided that they are all defined in terms of the same singularity exponent.

Great effort has been spent on investigating under which assumptions equality holds between some of the spectra, as a matter of fact mostly between spectra based on different scaling exponents. Indeed, the most interesting combinations seem to be $\dim(E^{[a]})$ with scaling exponents $h_k^{(n)}$ and $\alpha_k^{(n)}$, and $\tau^*(a)$ with scaling exponents $w_k^{(n)}$ and $\alpha_k^{(n)}$, the former for its importance in the analysis of regularity, the latter for its numerical relevance. It has become the accepted term in the literature to say that *the multifractal formalism holds* if any such spectra are equal; indeed they are in a generic sense [52]. However, this terminology might sometimes be confusing if the nature of the parts of such an equality is not indicated. We prefer here to call (1.14) *the multifractal formalism*: this formula 'holds' for any fixed choice of a singularity exponent as is shown in the paper.

1.5 Self-similarity and LRD

The statistical self-similarity as expressed in (1.1) makes fBm, or rather its increment process a paradigm of *long range dependence* (LRD). To be more explicit let δ denote some fixed lag and define *fractional Gaussian noise* (fGn) as

$$G(k) := B_H((k+1)\delta) - B_H(k\delta). \tag{1.15}$$

Having the LRD property means that the auto-correlation $r_G(k) := \mathbb{E}_\Omega[G(n+k)G(n)]$ decays so slowly that $\sum_k r_G(k) = \infty$. The presence of such strong dependence bears an important consequence on the aggregated processes

$$G^{(m)}(k) := \frac{1}{m} \sum_{i=km}^{(k+1)m-1} G(i). \tag{1.16}$$

They have a much higher variance, and variability, than would be the case for a short range dependent process. Indeed, if $X(k)$ are i.i.d., then $X^{(m)}(k)$ has variance $(1/m^2)\text{var}(X_0+\ldots+X_{m-1}) = (1/m)\text{var}(X)$. For G we find, due to (1.1) and $B_H(0) = 0$,

$$\text{var}(G^{(m)}(0)) = \text{var}\left(\frac{1}{m}B_H(m\delta)\right) = \text{var}\left(\frac{m^H}{m}B_H(\delta)\right) = m^{2H-2}\text{var}\left(G(0)\right). \tag{1.17}$$

For $H > 1/2$ this expression decays indeed much slower than $1/m$. As is shown in [19] $\text{var}(X^{(m)}) \simeq m^{2H-2}$ is equivalent to $r_X(k) \simeq k^{2H-2}$ and so, $G(k)$ is indeed LRD for $H > 1/2$ (this follows also directly from (7.3)).

Let us demonstrate with fGn how to relate LRD with multifractal analysis using only that it is a zero-mean processes, not (1.1). To this end let $\delta = 2^{-n}$ denote the finest resolution we will consider, and let 1 be the largest. For $m = 2^i$ $(0 \le i \le n)$ the process $mG^{(m)}(k)$ becomes simply $B_H((k+1)m\delta) - B_H(km\delta) = B_H((k+1)2^{i-n}) - B_H(k2^{i-n})$. But the second moment of this expression —which is also the variance— is exactly what determines $T_\alpha(2)$ (compare (1.10)). More precisely, using stationarity of G and substituting $m = 2^i$, we get

$$2^{-(n-i)T_\alpha(2)} \simeq \mathbb{E}_\Omega\left[S_\alpha^{n-i}(2)\right] = \sum_{k=0}^{2^{n-i}-1} \mathbb{E}_\Omega\left[|mG^{(m)}(k)|^2\right] = 2^{n-i}2^{2i}\text{var}\left(G^{(2^i)}\right). \tag{1.18}$$

This should be compared with the definition of the LRD-parameter H via

$$\text{var}(G^{(m)}) \simeq m^{2H-2} \qquad \text{or} \qquad \text{var}(G^{(2^i)}) = 2^{i(2H-2)}. \tag{1.19}$$

At this point a conceptual difficulty arises. Multifractal analysis is formulated in the limit of small scales ($i \to -\infty$) while LRD is a property at large scales ($i \to \infty$). Thus, the two exponents H and $T_\alpha(2)$ can in theory only be related when assuming that the scaling they represent is actually exact at all scales, and not only asymptotically. When this assumption is violated, the two approaches may provide strikingly different answers (compare Example 7.2).

In any real world application, however, one will determine both, H and $T_\alpha(2)$, by finding a *scaling region* $\underline{i} \le i \le \overline{i}$ in which (1.18) and (1.19) hold up to satisfactory precision. Comparing the two scaling laws in i yields $T_\alpha(2) + 1 - 2 = 2H - 2$, or

$$H = \frac{T_\alpha(2) + 1}{2}. \tag{1.20}$$

This formula expresses most pointedly, how *multifractal analysis goes beyond second order statistics*: in (1.26) we compute with $T(q)$ the scaling of *all* moments. The formula (1.20) is derived here for zero-mean processes, but can be put on more solid grounds using wavelet estimators of the LRD parameter [4] which are more robust than the ones obtained through variance of the increment process. The same formula (1.20) reappears also for multifractals, suggesting that it has some 'universal truth' to it, at least in the presence of 'perfect scaling' (see (1.29) and (7.25), but also Example 7.2).

1.6 Multifractal Processes

The most prominent examples where one finds coinciding, strictly concave multifractal spectra are the distribution functions of *cascade* measures [64, 56, 15, 33, 6, 82, 49, 91, 95, 86] for which $\dim(E^{[a]})$ and $T^*(a)$ are equal and have the form of a $\cap$ (see Figure 6 and also 3 (e)). These cascades are constructed through some multiplicative iteration scheme such as the binomial cascade, which is presented in detail in the paper with special emphasis on its wavelet decomposition. Having positive increments, however, this class of processes is sometimes too restrictive. fBm, as noted, has the disadvantage of a poor multifractal structure and does not contribute to a larger pool of stochastic processes with multifractal characteristics.

It is also notable that the first 'natural', truly multifractal stochastic process to be identified was Lévy motion [54]. This example is particularly appealing since scaling is not injected into the model by an iterative construction (this is what we mean by the term natural). However, its spectrum is, though it shows a non-trivial range of singularity exponents $H(t)$, degenerated in the sense that it is linear.

Construction and Simulation

With the formalism presented here, the stage is set for constructing and studying new classes of truly multi-fractional processes. The idea, to speak in Mandelbrot's own words, is inevitable after the fact. The ingredients are simple: a multifractal 'time warp', i.e., an increasing function or process $\mathcal{M}(t)$ for which the multifractal formalism is known to hold, and a function or process V with strong mono-fractal scaling properties such as *fractional Brownian motion* (fBm), a Weierstrass process or self-similar martingales such as Lévy motion. One then forms the compound process

$$\mathcal{V}(t) := V(\mathcal{M}(t)). \tag{1.21}$$

To build an intuition let us recall the method of *midpoint displacement* which can be used to define simple Brownian motion $B_{1/2}$ which we will also call *Wiener motion* (WM) for a clear distinction from fBm. This method constructs $B_{1/2}$ iteratively at dyadic points. Having constructed $B_{1/2}(k2^{-n})$ and $B_{1/2}((k+1)2^{-n})$ one defines $B_{1/2}((2k+1)2^{-n-1})$ as $(B_{1/2}(k2^{-n}) + B_{1/2}((k+1)2^{-n}))/2 + X_{k,n}$. The off-sets $X_{k,n}$ are independent zero mean Gaussian variables with variance such as to satisfy (1.1) with $H = 1/2$. Thus the name of the method. One way to obtain *Wiener motion in multifractal time* WM(MF) is then to keep the off-set variables $X_{k,n}$ as they are but to apply them at the time instances $t_{k,n}$ defined by $t_{k,n} = \mathcal{M}^{-1}(k2^{-n})$, i.e., $\mathcal{M}(t_{k,n}) = k2^{-n}$:

$$\mathcal{B}_{1/2}(t_{2k+1,n+1}) := \frac{\mathcal{B}_{1/2}(t_{k,n}) + \mathcal{B}_{1/2}(t_{k+1,n})}{2} + X_{k,n}. \tag{1.22}$$

This amounts to a *randomly located random displacement*, the location being determined by $\mathcal{M}$. Indeed, (1.21) is nothing but a time warp.

An alternative construction of 'warped Wiener motion' WM(MF) which yields an equally spaced sampling as opposed to the samples $\mathcal{B}_{1/2}(t_{k,n})$ provided by (1.22) is desirable. To this end, note first that the increments of WM(MF) become independent Gaussians once the path of $\mathcal{M}(t)$ is realized. To be more precise, fix n and let

$$\mathcal{G}(k) := \mathcal{B}((k+1)2^{-n}) - \mathcal{B}(k2^{-n}) = B_{1/2}(\mathcal{M}(k+1)2^{-n})) - B_{1/2}(\mathcal{M}(k2^{-n})). \tag{1.23}$$

For a sample path of $\mathcal{G}$ one starts by producing first the random variables $\mathcal{M}(k2^{-n})$. Once this is done, the $\mathcal{G}(k)$ simply are independent zero-mean Gaussian variables with variance $|\mathcal{M}(k+1)2^{-n})) - \mathcal{M}(k2^{-n})|$. This procedure has been used in Figure 3.

Global analysis

For the right hand side (RHS) of the multifractal formalism (1.14), we need only to know that V is an H-sssi process, meaning that the increment $V(t+u) - V(t)$ is equal in distribution to $u^H V(1)$ (compare (1.1)). Assuming independence between V and $\mathcal{M}$ a simple calculation reads as

$$\begin{aligned} &\mathbb{E}_\Omega \sum_{k=0}^{2^n-1} |\mathcal{V}((k+1)2^{-n}) - \mathcal{V}(k2^{-n})|^q \\ &= \sum_{k=0}^{2^n-1} \mathbb{E}\mathbb{E}\left[|V(\mathcal{M}((k+1)2^{-n})) - V(\mathcal{M}(k2^{-n}))|^q \,\middle|\, \mathcal{M}(k2^{-n}), \mathcal{M}((k+1)2^{-n}) \right] \\ &= \sum_{k=0}^{2^n-1} \mathbb{E}\left[|\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})|^{qH} \right] \mathbb{E}\left[|V(1)|^q \right]. \end{aligned} \tag{1.24}$$

Here, we dealt with increments $|\mathcal{V}((k+1)2^{-n}) - \mathcal{V}(k2^{-n})|$ for the ease of notation. With little more effort they can be replaced by suprema, i.e., by $2^{-nh_k^{(n)}}$, or even by $2^{-nw_k^{(n)}}$ for certain wavelet coefficients and under appropriate assumptions (see theorem 8.5). It follows, e.g., for $h_k^{(n)}$, that

$$\text{Warped } H\text{-sssi:} \qquad T_{h,\mathcal{V}}(q) = \begin{cases} T_{h,\mathcal{M}}(qH) & \text{if } \mathbb{E}_\Omega\left[|\sup_{0\le t\le 1} V(t)|^q \right] < \infty \\ -\infty & \text{else.} \end{cases} \tag{1.25}$$

Simple H-sssi process: When choosing the deterministic warp time $\mathcal{M}(t) = t$ we have $T_{\mathcal{M}}(q) = q - 1$ since $S_{\mathcal{M}}^{(n)}(q) = \text{const}2^n \cdot 2^{-nq}$ for all n. Also, $\mathcal{V} = V$. We obtain $T_{\mathcal{M}}(qH) = qH - 1$ which has to be inserted into (1.25) to obtain

$$\text{Simple } H\text{-sssi:} \qquad T_{h,V}(q) = \begin{cases} qH - 1 & \text{if } \mathbb{E}_\Omega\left[|\sup_{0\le t\le 1} V(t)|^q \right] < \infty \\ -\infty & \text{else.} \end{cases} \tag{1.26}$$

Local analysis of warped fBm

Let us now turn to the special case where V is fBm. Then, we use the term FB(MF) to abbreviate *fractional Brownian motion in multifractal time*: $\mathcal{B}(t) = B_H(\mathcal{M}(t))$. First, to obtain an intuition on what to expect from the spectra of $\mathcal{B}$ let us note that the moments appearing in (1.25) are finite for all q as we will see in lemma 7.4. Applying the Legendre transform yields easily that

$$T^*_{\mathcal{B}}(a) = \inf_q (qa - T_{\mathcal{B}}(q)) = \inf_q (qa - T_{\mathcal{M}}(qH)) = T^*_{\mathcal{M}}(a/H), \tag{1.27}$$

which is valid for all $a \in \mathbb{R}$ for which the second equality holds, i.e., for which the infimum is attained for q values in the range where $T_{\mathcal{B}}(q)$ is finite. In particular, for Brownian motion (fBm with $H = 1/2$) it holds for all a (compare lemma 7.4).

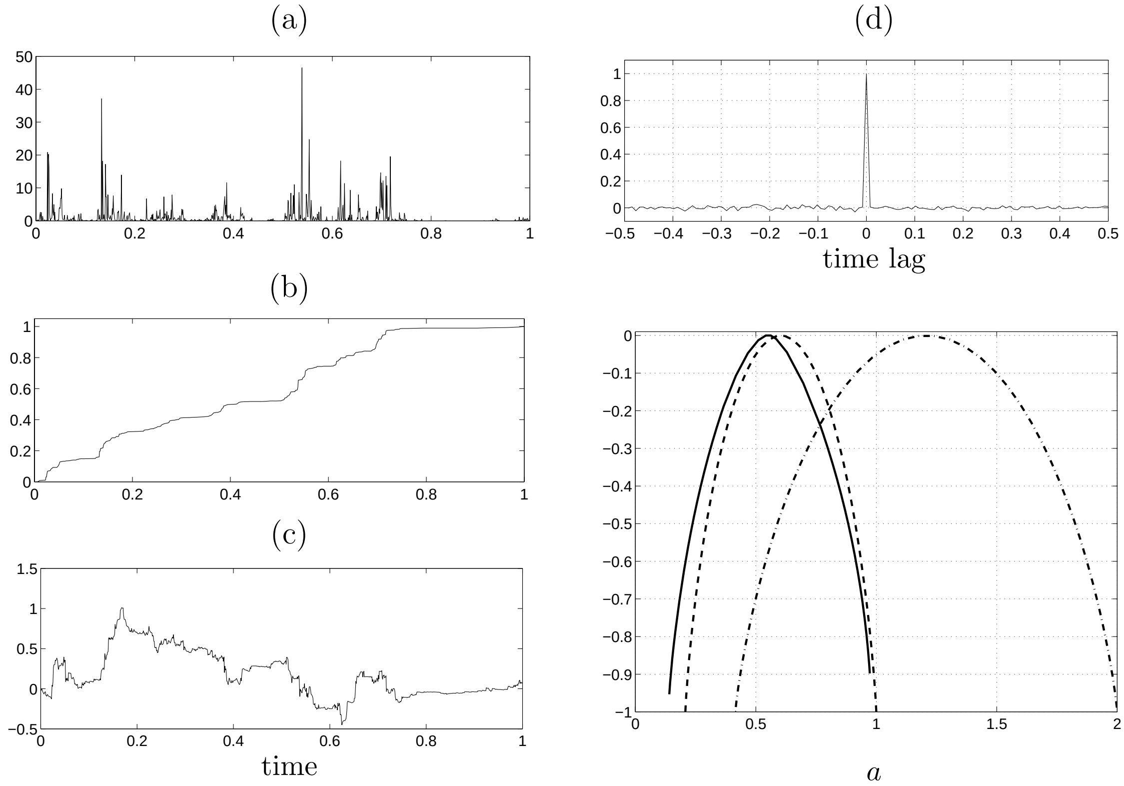

FIGURE 3. *Left: Simulation of Brownian motion in binomial time (a) Sampling of* $\mathcal{M}_b((k+1)2^{-n}) - \mathcal{M}_b(k2^{-n})$ *(*$k = 0, \ldots, 2^n - 1$*), indicating distortion of dyadic time intervals (b)* $\mathcal{M}_b((k2^{-n}))$*: the time warp (c) Brownian motion warped with (b):* $\mathcal{B}(k2^{-n}) = B_{1/2}(\mathcal{M}_b(k2^{-n}))$

Right: Estimation of $\dim(E^{[a]}_{\mathcal{B}})$ *via* $\tau^*_{w,\mathcal{B}}$ *(d) Empirical correlation of the Haar wavelet coefficients. (e) Dot-dashed:* $T^*_{\mathcal{M}_b}$ *(from theory), dashed:* $T^*_{\mathcal{B}}(a) = T^*_{\mathcal{M}_b}(a/H)$ *Solid: the estimator* $\tau^*_{w,\mathcal{B}}$ *obtained from (c). (Reproduced from [40].)*

Second, towards the local analysis we recall the uniform and strict Hölder continuity of the paths of fBm. In theorem 7.3 we state a precise result due to Adler [5] which reads roughly as

$$\sup_{|u|\le\delta} |\mathcal{B}(t+u)-\mathcal{B}(t)| = \sup_{|u|\le\delta} |B_H(\mathcal{M}(t+u))-B_H(\mathcal{M}(t))| \simeq \sup_{|u|\le\delta} |\mathcal{M}(t+u)-\mathcal{M}(t)|^H.$$

This is the key to conclude that B_H simply squeezes the Hölder regularity exponents by a factor H. Thus,

$$h_{\mathcal{B}}(t) = H\cdot h_{\mathcal{M}}(t), \qquad E_{\mathcal{M}}^{[a/H]} = E_{\mathcal{B}}^{[a]},$$

and, consequently, analogous to (1.27),

$$\boxed{\dim(E_{\mathcal{B}}^{[a]}) = \dim(E_{\mathcal{M}}^{[a/H]}).}$$

Figure 3 (d)-(e) displays an estimation of $\dim(E_{\mathcal{B}}^{[a]})$ using wavelets which agrees very closely with the form $\dim(E_{\mathcal{M}}^{[a/H]})$ predicted by theory. (For statistics on this estimator see [40, 41].)

Combining this with corollary 1.1 and (1.27) we may conclude:

Corollary 1.2 (Fractional Brownian Motion in Multifractal Time). *Let B_H denote fBm of Hurst parameter H. Let $\mathcal{M}(t)$ be of almost surely continuous paths and independent of B_H. Set $\mathcal{B}(t) = B_H(\mathcal{M}(t))$ and consider a multifractal analysis using $h_k^{(n)}$. Then, the* **multifractal warp formalism**

$$\boxed{\dim(E_{\mathcal{B}}^{[a]}) = f_{\mathcal{B}}(a) = \tau_{\mathcal{B}}^*(a) = T_{\mathcal{B}}^*(a) = T_{\mathcal{M}}^*(a/H)} \tag{1.28}$$

holds for any path and any a for which $\dim(E_{\mathcal{M}}^{[a/H]}) = T_{\mathcal{M}}^*(a/H) = T_{\mathcal{B}}^*(a)$.

The condition on a ensures that equality holds in the multifractal formalism for $\mathcal{M}$ and that the relevant moments are finite so that (1.27) holds. If satisfied, then the local, or fine, multifractal structure of $\mathcal{B}$ captured in $\dim(E_{\mathcal{B}}^{[a]})$ on the left side in (1.28) can be estimated through grain based, simpler and numerically more robust spectra on the right side, such as $\tau_{\mathcal{B}}^*(a)$ (compare Figure 3 (e)).

Moreover, the 'warp formula' (1.28) is appealing since it allows to *separate* the LRD parameter of fBm and the multifractal spectrum of the time change $\mathcal{M}$. Indeed, provided that $\mathcal{M}$ is almost surely increasing one has $T_{\mathcal{M}}(1) = 0$ since $S^{(n)}(0) = \mathcal{M}(1)$ for all n. Thus, $T_{\mathcal{B}}(1/H) = 0$ exposes the value of H. Alternatively, the tangent at $T_{\mathcal{B}}^*$ through the origin has slope $1/H$. Once H is known $T_{\mathcal{M}}^*$ follows easily from $T_{\mathcal{B}}^*$.

Simple fBm: When choosing the deterministic warp time $\mathcal{M}(t) = t$ we have $\mathcal{B} = B_H$ and $T_{B_H}(q) = qH-1$ as in (1.26). In the special case of Brownian motion ($H = 1/2$) we may apply (1.28) for all a showing that all $h_k^{(n)}$-based spectra consist of the point $(H, 1)$ only. This makes the mono-fractal character of this process most explicit. In general, however, artifacts which are due mainly to diverging moments may distort this simple picture (see Section 7.3).

LRD and estimation of warped Brownian motion

Let $\mathcal{G}(k) := \mathcal{B}((k+1)2^{-n}) - \mathcal{B}(k2^{-n})$ be 'fGn in multifractal time' (see (1.23) for the case $H = 1/2$). Calculating auto-correlations explicitly, lemma 8.8 shows that $\mathcal{G}$ is second order stationary provided $\mathcal{M}$ has stationary increments. Assuming $\mathbb{E}[\mathcal{M}(s)^{2H}] = \text{const} \cdot s^{T(2H)+1}$, the correlation of $\mathcal{G}$ is of the form of ordinary fGn, but decaying as $r_{\mathcal{G}}(k) \simeq k^{2H_{\mathcal{G}}-2}$ where

$$H_{\mathcal{G}} = \frac{T_{\mathcal{M}}(2H) + 1}{2}. \tag{1.29}$$

Let us discuss some special cases. An obvious choice for a subordinator $\mathcal{M}$ is Lévy motion, an H'-self-similar, $1/H'$-stable process. It has independent, stationary increments. Since the relation (1.1) holds with H' as the scaling parameter, we have $T(q) = qH' - 1$ from (1.26). Moreover, $\mathcal{M}(s)^{2H}$ is equal in distribution to $(s^{H'}\mathcal{M}(1))^{2H}$ and indeed $\mathbb{E}[\mathcal{M}(s)^{2H}] = \text{const} \cdot s^{2HH'} = \text{const} \cdot s^{T(2H)+1}$. This expression is finite for $2H < 1/H'$. In summary, $H_{\mathcal{G}} = HH' < 1/2$.

For a continuous, increasing warp time $\mathcal{M}$, on the other hand, we have always $T_{\mathcal{M}}(0) = -1$ and $T_{\mathcal{M}}(1) = 0$. (Lévy motion is discontinuous; it is increasing for $H' < 1$, in which case $T(1)$ is not defined.) Exploiting the concave shape of $T_{\mathcal{M}}$ we find that $H < H_{\mathcal{G}} < 1/2$ for $0 < H < 1/2$, and $1/2 < H_{\mathcal{G}} < H$ for the LRD case $1/2 < H < 1$.

Especially in the case of $H = 1/2$ ('white noise in multifractal time') $\mathcal{G}(k)$ becomes *uncorrelated* (see also (8.20)). Notably, this is a stronger statement than the observation that the $\mathcal{G}(k)$ are *independent conditioned* on $\mathcal{M}$ (compare Section 1.6). As a particular consequence, wavelet coefficients will decorrelate fast for the compound process $\mathcal{G}$, not only when conditioning on $\mathcal{M}$ (compare Figure 3 (d)). This is favorable for estimation purposes as it reduces the error variance. Finally, for increasing $\mathcal{M}$ we have $T(1) = 0$ and the requirements for (1.29) reduce to the simple $\mathbb{E}[\mathcal{M}(s)] = s$. Multiplicative processes with this property (as well as stationary increments) have been introduced recently [14, 70, 74, 105].

Though seemingly obvious it should be pointed out that the vanishing correlations of $\mathcal{G}$ in the case $H = 1/2$ should not be taken as an indication of independence. After all, $\mathcal{G}$ becomes Gaussian only when conditioning on knowing $\mathcal{M}$. A strong, higher order dependence in $\mathcal{G}$ is hidden in the dependence of the increments of $\mathcal{M}$ which determine the variance of $\mathcal{G}(k)$ as in (1.23). Indeed, Figure 3 (c) shows clear phases of monotony of $\mathcal{B}$ indicating positive dependence in its increments $\mathcal{G}$, despite vanishing correlations. Mandelbrot calls this the 'blind spot of spectral analysis'.

Multifractals in multifractal time

Despite of its simplicity the presented scheme for constructing multifractal processes allows for various play-forms some of which are little explored. First of all, for simulation purposes one might subject the *randomized Weierstrass-Mandelbrot function* to time change rather than fBm itself.

Next, we may choose to replace fBm with a more general self-similar process (7.1) such as Lévy motion. Difficulties arise here since Lévy motion is itself a multifractal with varying Hölder regularity and the challenge lies in studying which exponents of the 'multifractal time' and the motion are most likely to meet. A solution for the spectrum $f(a)$ is given in corollary 8.13 which actually applies to arbitrary processes Y with stationary increments (compare theorem 8.15) replacing fBm. In its most compact form our result reads as:

Corollary 1.3 (Lévy motion in multifractal time). *Let L_H denote Lévy stable motion and let $\mathcal{M}$ be a binomial cascade (see 5.1) independent of L_H and set $\mathcal{V}(t) = L_H(\mathcal{M}(t))$. Then, for almost all paths*

$$f_{\mathcal{V}}(a) = \tau^*_{\mathcal{V}}(a) \overset{\text{a.s.}}{=} T^*_{\mathcal{V}}(a) \tag{1.30}$$

for all α where $T^*_{\mathcal{V}} > 0$. The envelope $T^*_{\mathcal{V}}$ can be computed through the **warp formula**

$$\boxed{T_{\mathcal{V}}(q) = T_{\mathcal{M}}\Big(T_{L_H}(q) + 1\Big)\ .} \tag{1.31}$$

Recall (1.26) for a formula of T_{L_H}, which is generalized in (7.10). As the paper shows (1.30) and (1.31) hold actually in more generality.

Finally, for $\mathcal{Y}(t) = Y(\mathcal{M}(t))$ where Y and $\mathcal{M}$ are both almost surely increasing, i.e., multifractals in the classical sense, a close connection to the so-called 'relative multifractal analysis' [95] can be established using the concept of inverse multifractals [94]: understanding the multifractal structure of $\mathcal{Y}$ is equivalent to knowing the multifractal spectra of Y with respect to $\mathcal{M}^{-1}$, the inverse function of $\mathcal{M}$. We will show how this can be resolved in the simple case of binomial cascades.

2 Singularity Exponents

For simplicity we consider processes Y over a probability space $(\Omega, \mathcal{F}, P_\Omega)$ and defined on a compact interval, which we assume without loss of generality to be $[0, 1]$. Generalization to higher dimensions is straightforward and extending to processes defined on $\mathbb{R}$ is simple and will be indicated.

2.1 Hölder Continuity

As pointed out in the introduction, the erratic behavior of a continuous process $Y(t)$ maybe indicative of crucial properties with relevance in various applications. This local behavior of Y at a given time t can be characterized to a first approximation by comparison with an algebraic function as follows:

Definition 2.1. *A function or the path of a process Y is said to be in C^h_t if there is a polynomial P_t such that*

$$|Y(u) - P_t(u)| \le C|u - t|^h$$

for u sufficiently close to t. Then, the degree of local Hölder regularity *of Y at t is*

$$H(t) := \sup\{h : Y \in C^h_t\}. \tag{2.1}$$

As usual, let $\lfloor x \rfloor$ denote the largest integer smaller or equal to x. If the Taylor polynomial of degree $\lfloor H(t) \rfloor$ exists, then P is necessarily that Taylor polynomial. As the example $Y(t) = 1 + t + t^2 + t^{3.5}\sin(1/t)$ shows, P may be different from the Taylor polynomial if Y does not have sufficient degree of smoothness. Here, $Y(t)$ has only one derivative at $t = 0$, and its Taylor polynomial at $t = 0$ is $u \mapsto 1 + u$ while $P_0(u) = 1 + u + u^2$.

Of special interest for our purpose is the case when the approximating polynomial P_t is a constant, i.e., $P_t(u) = Y(t)$, in which case $H(t)$ can be computed easily. To this end:

Definition 2.2. *Let us agree on the convention* $\log(0) = -\infty$ *and set*

$$h(t) := \liminf_{\varepsilon \to 0} \frac{1}{\log_2(2\varepsilon)} \log_2 \sup_{|u-t|<\varepsilon} |Y(u) - Y(t)|. \tag{2.2}$$

Note first that for any $h < h(t)$ we have $|Y(u) - Y(t)| \leq C|u - t|^h$, thus $Y \in C_t^h$. We conclude that always

$$h(t) \leq H(t). \tag{2.3}$$

It is easy to see that if P_t is *known* to be a constant then $H(t) = h(t)$. On the other hand, we show:

Lemma 2.3. *If* $h(t) \notin \mathbb{N}$ *then* P_t *is a constant, and* $h(t) = H(t)$.

As the example $Y(t) = t^2 + t^{2.4}$ shows the conclusion does not necessarily hold when $h(t) \in \mathbb{N}$.

Proof

We will show the dual statement: If $H(t) > h(t)$ then $h(t)$ must be an integer. Together with (2.3) this will certainly establish the lemma.

So, we assume $H(t) > h(t)$. Then there is $h > h(t)$ and a polynomial $P_t(\cdot)$ such that $|Y(u) - P_t(u)| \leq C|u - t|^h$. Note that P_t cannot be a constant: if it were constant, then $h \leq h(t)$ due to the very definition of $h(t)$. Thus, we may write the error-minimizing polynomial as $P_t(u) = Y(t) + (u - t)^m \cdot Q(u)$ for some integer $m \geq 1$ and some polynomial Q without zero at t. Assume first that $m < h(t)$ and choose h' such that $m < h' < h(t)$. Writing $Y(u) - P_t(u) = (Y(u) - Y(t)) - (P_t(u) - Y(t))$, the first term is smaller than $|u - t|^{h'}$ and the second term, decaying as $C|u - t|^m$, governs. Whence $h = m < h(t)$, a contradiction against the assumption $h > h(t)$. Assuming $m > h(t)$ choose h' such that $m > h' > h(t)$ and a sequence u_n such that $|Y(u_n) - Y(t)| \geq |u_n - t|^{h'}$. Then, $|Y(u_n) - P_t(u_n)| \geq (1/2)|u_n - t|^{h'}$ for large n and $h \leq h'$. Letting $h' \to h(t)$ we get again a contradiction. We conclude that $h(t)$ equals m. $\diamondsuit$

An essential simplification for both, analytical and empirical study, is to replace the continuous limit in (2.2) by a discrete one. To this end we introduce some notation

Definition 2.4. *Let* $k_n(t) := \lfloor t 2^n \rfloor$. *Then,* $k_n(t)$ *is the unique integer such that*

$$t \in I_{k_n}^{(n)} := [k_n 2^{-n}, (k_n + 1)2^{-n}[. \tag{2.4}$$

As n increases the intervals $I_k^{(n)}$ form a nested decreasing sequence (compare Figure 4). Now, when defining a discrete approximation to $h(t)$ we have to imitate in a discrete manner a *ball* around t over which we will consider the increments of Y. Accounting for the fact that t could lay very close to the boundary of $I_{k_n}^{(n)}$, we set

Definition 2.5. *The* coarse Hölder exponents *of* Y *are*

$$h_{k_n}^{(n)} := -\frac{1}{n} \log_2 \sup \left\{ |Y(u) - Y(t)| \,:\, u \in [(k_n - 1)2^{-n}, (k_n + 2)2^{-n}[\right\}. \tag{2.5}$$

To compare the limiting behavior of these exponents with $h(t)$ we choose n such that $2^{-n+1} \leq \varepsilon < 2^{-n+2}$. We have

$$[(k_n - 1)2^{-n}, (k_n + 2)2^{-n}[\subset [t + \varepsilon, t - \varepsilon[\subset [(k_{n-2} - 1)2^{-n+2}, (k_{n-2} + 2)2^{-n+2}[.$$

from which it follows immediately that

Lemma 2.6.

$$h(t) = \liminf_{n \to \infty} h_{k_n}^{(n)}$$

It is essential to note that the countable set of numbers $h_{k_n}^{(n)}$ contains all the scaling information of interest to us. Being defined pathwise, they are random variables.

2.2 Scaling of Wavelet Coefficients

The discrete wavelet transform represents a 1-d process $Y(t)$ in terms of shifted and dilated versions of a prototype bandpass wavelet function $\psi(t)$, and shifted versions of a low-pass scaling function $\phi(t)$ [23, 106]. Made precise in the vocabulary of Hilbert spaces: For special choices of the wavelet and scaling functions, the atoms

$$\psi_{j,k}(t) := 2^{j/2}\, \psi\big(2^j t - k\big)\,, \quad \phi_{j,k}(t) := 2^{j/2}\, \phi\big(2^j t - k\big)\,, \qquad j, k \in \mathbb{Z} \tag{2.6}$$

form an orthonormal basis and we have the representations [23, 106]

$$Y(t) = \sum_k D_{J_0,k}\, \phi_{J_0,k}(t) \; + \; \sum_{j=J_0}^{\infty} \sum_k C_{j,k}\, \psi_{j,k}(t), \tag{2.7}$$

with

$$C_{j,k} := \int Y(t)\, \psi_{j,k}(t)\, \mathrm{d}t, \qquad D_{j,k} := \int Y(t)\, \phi_{j,k}(t)\, \mathrm{d}t. \tag{2.8}$$

For a wavelet $\psi(t)$ centered at time zero and frequency f_0, the *wavelet coefficient* $C_{j,k}$ measures the signal content around time $2^{-j}k$ and frequency $2^j f_0$. The *scaling* coefficient $D_{j,k}$ measures the local mean around time $2^{-j}k$. In the wavelet transform, j indexes the *scale* of analysis: J_0 can be chosen freely and indicates the coarsest scale or lowest resolution available in the representation.

Compactly supported wavelets are of especial interest in this paper (see [23]. The Haar scaling and wavelet functions (see Figure 4(a)) provide the simplest example of such an orthonormal wavelet basis: ϕ is the indicator function of the unit interval, while $\psi(t) = \phi(2t) - \phi(2t-1)$. For a process supported on the unit interval one may, thus, choose $J_0 = 0$. The supports of the fine-scale scaling functions nest inside the supports of those at coarser scales; this can be neatly represented by the binary tree structure of Figure 4(b). Row (scale) j of this scaling coefficient tree contains an approximation to $Y(t)$ of resolution 2^{-j}. Row j of the complementary wavelet coefficient tree (not shown) contains the details in scale $j+1$ of the scaling coefficient tree that are suppressed in scale j. In fact, for the Haar wavelet we have

$$\begin{aligned} D_{j,k} &= 2^{-1/2}(D_{j+1,2k} + D_{j+1,2k+1}), \\ C_{j,k} &= 2^{-1/2}(D_{j+1,2k} - D_{j+1,2k+1}). \end{aligned} \tag{2.9}$$

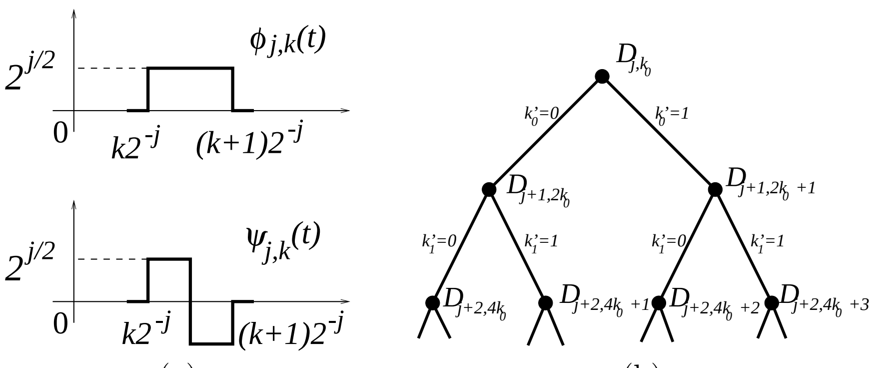

FIGURE 4. *(a) The Haar scaling and wavelet functions $\phi_{j,k}(t)$ and $\psi_{j,k}(t)$. (b) Binary tree of scaling coefficients from coarse to fine scales.*

Wavelet decompositions contain considerable information on the singularities of a process Y. Indeed, adapting the argument of [51, p. 291] (note the L^2 wavelet normalization used here — as opposed to L^1 in [51]) we find

Lemma 2.7. *Fix t and let $k_n = k_n(t)$ as in (2.4). If $|Y(s) - Y(t)| = O(|s-t|^h)$ for $s \to t$, and if ψ is a compactly supported function with $\int \psi = 0$ and $\int |\psi| < \infty$, then*

$$2^{n/2}\,|C_{n,k_n}| = 2^n \left| \int Y(s)\,\psi(2^n s - k)\;ds \right| = O\big(2^{-nh}\big) \qquad \textit{for } n \to \infty. \tag{2.10}$$

Proof
The compact support of ψ is assumed only for simplicity of the argument (compare [51]), and we take it to be $[0,1]$. Then, $\psi(2^n s - k) = 0$ for $s \notin I_k^{(n)}$. Also, $|s-t| \le 2^{-n}$ for all $s \in I_{k_n}^{(n)}$. These facts, together with $\int Y(t)\psi(s)ds = 0$ and the given estimate on

Y allow to conclude as

$$\begin{aligned}2^{-n/2}\,|C_{n,k_n}| &= \left|\int_{I_k^{(n)}}(Y(s)-Y(t))\,\psi(2^n s-k)\,\mathrm{d}s\right| \\ &\le C\cdot\int_{I_k^{(n)}}|s-t|^h\,|\psi(2^n s-k)\,|\,\mathrm{d}s \\ &= C\cdot 2^{-nh}\int_{I_k^{(n)}}|\psi(2^n s-k)\,|\,\mathrm{d}s = C\cdot 2^{-nh}2^{-n}\cdot\int_{\mathbb{R}}|\psi(s)\,|\,\mathrm{d}s\end{aligned}$$

◇

Following the proof of [51, p. 291] more closely it is easily seen that the assumption of ψ being compactly supported is not really needed. Indeed, a fast decay is sufficient and the result holds also for functions which don't necessarily form a basis such as derivatives of the Gaussian $\exp(-x^2)$. To distinguish such functions from the orthogonal wavelets we will address them as 'analyzing wavelets'.

In order to invert lemma 2.7 and infer the Hölder regularity of Y from the decay of wavelet coefficients one needs the representation (2.7), sufficient wavelet regularity as well as some knowledge on the decay of the maximum of the wavelet coefficients in the vicinity of t, as developed in the pioneering work of [78] (see also [51] and [23, Thm. 9.2]).

All this suggests that the left hand side of (2.10) could produce an alternative useful description of the local behavior oscillatory behavior of Y.

Definition 2.8. *The* coarse wavelet singularity exponents *of Y are*

$$w_{k_n}^{(n)} := -\frac{1}{n}\log_2\left|2^{n/2}C_{n,k_n}\right|. \tag{2.11}$$

The local singularity exponent of wavelet coefficients is then $w(t) := \liminf_{n\to\infty} w_{k_n}^{(n)}$.

Indeed, while the coarse Hölder exponents $h_k^{(n)}$ give the exact Hölder regularity but only under the assumption of constant approximating polynomials, using wavelets has the advantage of yielding an analysis which is largely unaffected by polynomial trends in Y. This is due to vanishing moments $\int t^m\psi(t)\mathrm{d}t = 0$ which are typically built into wavelets [23]. However, for a reliable estimation of true Hölder continuity through wavelets one has to employ the lines of maxima, a method pioneered convincingly in [78] (see also [51, 23, 53, 7]).

In any case, the decay of wavelet coefficients is interesting in itself as it relates to LRD (compare [4] and Section 7.4) and regularity spaces such as Besov spaces [89, 30].

2.3 Other Singularity Exponents

The 'classical' multifractal analysis of a singular measure μ on $[0,1]$, translated into the notations used here, has always been concerned with the study of the singularity structure of its primitive

$$\mathcal{M}(t) = \int_0^t \mu(\mathrm{d}s) = \mu([0,t]), \tag{2.12}$$

which is an almost surely increasing process. Consequently, the supremum in $h_{k_n}^{(n)}$ simplifies to $|\mathcal{M}((k_n+2)2^{-n}) - \mathcal{M}((k_n-1)2^{-n})|$ and one is lead to introduce yet another singularity exponent:

Definition 2.9. *The* coarse increment exponents *of* $\mathcal{M}$ *are*

$$\alpha_{k_n}^{(n)} := -\frac{1}{n}\log_2 |\mathcal{M}\big((k_n+1)2^{-n}\big) - \mathcal{M}(k_n 2^{-n})| = -\frac{1}{n}\log_2 \mu\big(I_{k_n}^{(n)}\big). \tag{2.13}$$

The local singularity exponent of increments is then $\alpha(t) := \liminf_{n\to\infty} \alpha_{k_n}^{(n)}$.

As we will elaborate in lemma 5.5 the increment and Hölder exponents $\alpha_k^{(n)}$ and $h_k^{(n)}$ provide largely the same analysis for increasing $\mathcal{M}$, however, there is a crucial lesson to be learned for measures μ with fractal support in Section 5.6.

For further examples of singularity exponents we would like to refer to [63] which treats the case of exponents which are so-called Choquet capacities, a notion which is not needed to develop the multifractal formalism as we will show in this paper.

Also, [87] considers an arbitrary function $\xi(I)$ from the space of all intervals to $\mathbb{R}^+$ (instead of only the $I_k^{(n)}$) and develops a multifractal formalism similar to the one presented here. There, it is suggested to consider the oscillations of Y around the mean, i.e.,

$$\xi(I) := \int_I \left| Y(t) - \frac{\int_I Y(s)\mathrm{d}s}{|I|} \right| \mathrm{d}t \tag{2.14}$$

This gives raise to the singularity exponent $-(1/n)\log_2(\xi(I_k^{(n)}))$ which is of particular interest since it can be used to define oscillation spaces such as Sobolev spaces and Besov spaces. Alternatively, interpolating Y in the interval I by the linear function $a_I + b_I t$, one could use

$$\xi(I) := \left(\int_I (Y(t) - (a_I + b_I t))^2 \,\mathrm{d}t \right)^{1/2}. \tag{2.15}$$

This $\xi(I)$ measures the variability of Y and is related to the dimension of the paths of Y. In the definitions (2.14) and (2.15) constant, resp. linear terms are subtracted from Y. This may remind one of the use of wavelets with one, resp. two vanishing moments.

In conclusion, there are various useful notions of singularity exponents which may provide a characterization of a process Y of relevance in particular applications. Being aware that these descriptions may very well differ for the same Y according to one's choice of an exponent, we do not attempt to value one over the other, but rather present some aspects of multifractal analysis which are valid for any such choice, i.e., a form of the celebrated multifractal formalism.

3 Multifractal Analysis

Multifractal analysis has been discovered in the context of fully developed turbulence [64, 36, 42, 46, 45] and subsequently further developed in physical and mathematical circles (see [15, 56, 13, 12, 16, 102, 9, 66, 49, 78, 94, 22, 58, 33, 6, 53, 82, 25, 86, 8, 54, 31, 74] to give only a short list of some relevant work done in this area). At the beginning stands the discovery that on fractals local scaling behavior as measured by exponents $h_k^{(n)}$, $\alpha_k^{(n)}$ or $w_k^{(n)}$, is not uniform in general. In other words, $h(t)$, $\alpha(t)$ and $w(t)$ are typically not constant in t but assume a whole range of values, thus imprinting a rich structure on the object of interest [69, 80, 2, 88]. This structure can be characterized either in geometrical terms making use of the concept of dimension, or in statistical terms based on sample moments. A tight connection between these two descriptions emerges from the *multifractal formalism*.

As we will see, as far as the validity of the multifractal formalism is concerned there is no restriction in choosing a singularity exponent which seem fit for describing scaling behavior of interest, as long as one is consistent in using the same exponents for both, the geometrical and the statistical description. To express this fact we consider in this section the arbitrary coarse singularity exponent

$$s_k^{(n)} \qquad (k = 0, \ldots, 2^n - 1,\, n \in \mathbb{N}), \tag{3.1}$$

which may be *any* sequence of random variables. To keep a connection with what was said before think of $s_k^{(n)}$ as representing a coarse singularity exponent related to the oscillations of Y over the dyadic interval $I_k^{(n)}$. To accommodate processes which are constant over some intervals we explicitly allow $s_k^{(n)}$ to take the value ∞.

3.1 Dimension based Spectra

The strongest interests of the mathematical community are in the various measure theoretical dimensions of sets $E^{[a]}$ which are defined pathwise in terms of limiting behavior of $s_{k_n}^{(n)}$ as $n \to \infty$, as follows

$$E^{[a]} := \{t \,:\, \liminf_{n\to\infty} s_{k_n}^{(n)} = a\},$$

$$K^{[a]} := \{t \,:\, \lim_{n\to\infty} s_{k_n}^{(n)} = a\} \tag{3.2}$$

These sets are typically 'fractal' meaning loosely that they have a complicated geometric structure and more precisely that their dimensions are non-integer. A compact description of the singularity structure of Y is, therefore, in terms of the dimensions of $E^{[a]}$ and $K^{[a]}$.

Definition 3.1. *The* Hausdorff spectrum *is the function*

$$a \mapsto \dim(E^{[a]}), \tag{3.3}$$

where $\dim(E)$ *denotes the Hausdorff dimension of the set E [103].*

The sets $E^{[a]}$ ($a \in \mathbb{R}$) form a *multifractal decomposition* of the support of Y, i.e., they are disjoint and their union is the support of Y. We will loosely address Y as a *multifractal* if this decomposition is rich, i.e. if the sets $E^{[a]}$ ($a \in \mathbb{R}$) are highly interwoven or even dense in the support of Y.

We should point out that the study of singular measures (deterministic and random) has often focussed on the simpler sets $K^{[a]}$ and their spectrum $\dim(K^{[a]})$ [56, 15, 33, 6, 82, 94, 91, 93, 8]. However, lemma 3.3 which is established below allows to extend most of these results in order to provide formulas for $\dim(E^{[a]})$ also.

3.2 Grain based Spectra

An alternative to the above geometrical description of the singularity structure relies on counting:‡

$$N^{(n)}(a,\varepsilon) := \#\{k = 0,\dots,2^n - 1 \,:\, a - \varepsilon \le s_k^{(n)} < a + \varepsilon\}. \tag{3.4}$$

Note, that infinite $s_k^{(n)}$ have no influence here. Indeed, computing multifractal spectra at $a = \infty$ requires usually special attention (see [94]).

Definition 3.2. *The* grain based spectrum *is the function*

$$f(a) \;:=\; \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \log_2 N^{(n)}(a,\varepsilon). \tag{3.5}$$

To establish some of the almost sure pathwise properties it is convenient to introduce also

$$\underline{f}(a) \;:=\; \lim_{\varepsilon\downarrow 0} \liminf_{n\to\infty} \frac{1}{n} \log_2 N^{(n)}(a,\varepsilon) \tag{3.6}$$

This approach has grown out of the difficulties involved with computation of Hausdorff dimensions, in particular in any real world applications. Using a mesh of given grain size as in (3.4) instead of arbitrary coverings as in $\dim(E^{[a]})$ leads generally to more simple notions. However, f should not be regarded as an auxiliary vehicle but rather meriting attention in its own right. This point was already made in Section 1.2, and we hope to make it stronger as we proceed in our presentation.

Our first remark on $f(a)$ concerns the fact that the counting used in its definition, i.e., $N^{(n)}(a,\varepsilon)$ may be used to estimate box dimensions. Based on this fact it was shown in [94] that

$$\dim(K^{[a]}) \le f(a). \tag{3.7}$$

Note that the set in (3.7) is a subset of $E^{[a]}$ since in its points the sequence of $s_k^{(n)}$ is actually required to converge. Also, we will later need a lower bound on $\underline{f}(a)$. Therefore, we provide two formulas that are sharper than (3.7).

‡More generally, using c-ary intervals in Euclidean space $\mathbb{R}^d$ k_n will range from 0 to $c^{nd}-1$. Logarithms will have to be taken to the base c since we seek the asymptotics of $N^{(n)}(a,\varepsilon)$ in terms of a powerlaw of resolution at stage n, i.e., $N^{(n)}(a,\varepsilon) \simeq c^{nf(a)}$. The maximum value of $f(a)$ will be d.

Lemma 3.3.

$$\dim(E^{[a]}) \le f(a) \tag{3.8}$$

and

$$\dim(K^{[a]}) \le \underline{f}(a). \tag{3.9}$$

Proof

Fix a. To prove the first part of the lemma consider an arbitrary $\gamma > f(a)$, and choose $\eta > 0$ such that $\gamma > f(a) + 2\eta$. Then, there is $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ such that

$$N^{(n)}(a,\varepsilon) \le 2^{n(f(a)+\eta)}$$

for all $n > m_0$. Let us define $J(m) := \cup\{k_n \,:\, n \ge m \text{ and } a-\varepsilon \le s^{(n)}_{k_n} \le a+\varepsilon\}$. Then, for any m the intervals $I^{(n)}_{k_n}$ with $k_n \in J(m)$ form a cover of $E^{[a]}$. These intervals are of length less than 2^{-m}. Moreover, for any $m > m_0$ we have

$$\sum_{k_n \in J(m)} |I^{(n)}_{k_n}|^\gamma = \sum_{n \ge m} N^{(n)}(a,\varepsilon)\cdot 2^{-n\gamma} \le \sum_{n\ge m} 2^{-n(\gamma - f(a) - \eta)} \le \sum_{n \ge m} 2^{-n\eta}$$

tends to zero with $m \to \infty$. We conclude that the γ-dimensional Hausdorff measure of $E^{[a]}$ is zero, hence, $\dim(E^{[a]}) \le \gamma$. Letting $\gamma \to f(a)$ completes the first part.

Aiming at using $\underline{f}(a)$ for an estimate of Hausdorff dimensions consider an arbitrary $\gamma > \underline{f}(a)$, and choose $\eta > 0$ such that $\gamma > \underline{f}(a) + 2\eta$. Note that $N^{(n)}(a,\varepsilon) \le 2^{n(\underline{f}(a)+\eta)}$ similar as before, but this time only for some infinitely many indices n (not *all* large n). This is very little information and not sufficient to tackle $E^{[a]}$; even to deal with $K^{[a]}$ we need the auxiliary sets $K_l := \{t \,:\, a-\varepsilon < s^{(n)}_k < a+\varepsilon \text{ for } k = k_n(t) \text{ and all } n > l\}$ (this approach is similar to [94, p. 137]). An efficient cover of K_l is provided by the sets $I^{(n)}_k$ where $n > l$ is fixed and where k satisfies $a - \varepsilon < s^{(n)}_k < a + \varepsilon$. We find

$$\sum_{k:\, a-\varepsilon < s^{(n)}_k < a+\varepsilon} |I^{(n)}_k|^\gamma = N^{(n)}(a,\varepsilon)\cdot 2^{-n\gamma} \le 2^{-n(\gamma - \underline{f}(a) - \eta)},$$

where the last inequality holds at least for some infinitely many indices n. This allows to conclude that the γ-dimensional Hausdorff measure of K_l is zero and $\dim(K_l) \le \gamma$. Since $K^{[a]} = \cup_l K_l$ we conclude $\dim(K^{[a]}) \le \gamma$ by countable continuity of the Hausdorff measure $\dim(\cdot)$ [32, p. 29]. Letting $\gamma \to \underline{f}(a)$ completes the proof. ♢

3.3 Partition Function and Legendre Spectrum

The second note on the grain spectrum $f(a)$ concerns its interpretation as a Large Deviation Principle (LDP). To this end we consider $N^{(n)}(a,\varepsilon)/2^n$ to be the probability to find (for a fixed realization of Y) a number $k_n \in \{0,\dots,2^n-1\}$ such that $s^{(n)}_{k_n} \in [a-\varepsilon, a+\varepsilon]$. Typically, there will be one "expected" or most frequent value of $\lim_{n\to\infty} s^{(n)}_{k_n}$, denoted $\hat{a}$, and $f(a)$ will reach its maximum 1 at $a = \hat{a}$. If a differs from $\hat{a}$, on the other hand, then $[a-\varepsilon, a+\varepsilon]$ will not contain $\hat{a}$ for small ε and the chance to observe coarse exponents $s_{k_n}^{(n)}$ which lie in $[a-\varepsilon, a+\varepsilon]$ will decrease exponentially fast with rate given by $f(a)$.

Appealing to the theory of LDP-s we consider the random variable $A_n = -ns_K^{(n)}\ln(2)$ where K is randomly picked from $\{0,\ldots,2^n-1\}$ with uniform distribution U_n (recall that the we study one fixed realization or path of Y) and introduce its 'logarithmic moment generating function':

Definition 3.4. *The* partition function *of a path of Y is defined for all $q \in \mathbb{R}$ as*

$$\tau(q) := \liminf_{n\to\infty} -\frac{1}{n}\log_2 S^{(n)}(q), \tag{3.10}$$

where

$$S^{(n)}(q) := \sum_{k=0}^{2^n-1} \exp\left(-qns_k^{(n)}\ln(2)\right) = \sum_{k=0}^{2^n-1} 2^{-nqs_k^{(n)}} = 2^n \mathbb{E}_n\left[2^{-nqs_k^{(n)}}\right]. \tag{3.11}$$

Here, $\mathbb{E}_n$ stands for expectation with respect to U_n. To avoid trivialities we set $2^{-q\infty} := 0$ for all $q \in \mathbb{R}$, *i.e., infinite $s_k^{(n)}$ give no contribution to the partition sum.*

Theorems on LDP such as the one of Gärtner-Ellis [27] apply then to yield the following result which was established in a slightly stronger version in [91]:

Theorem 3.5. *If the limit*

$$\tau(q) = \lim_{n\to\infty} -\frac{1}{n}\log_2 S^{(n)}(q) \tag{3.12}$$

exists and is finite for all $q \in \mathbb{R}$, and if $\tau(q)$ is a differentiable function of q, then the double limit

$$f(a) = \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} \frac{1}{n}\log_2 N^{(n)}(a,\varepsilon) \tag{3.13}$$

exists, in particular $f(a) = \underline{f}(a)$, and

$$f(a) = \tau^*(a) := \inf_{q\in\mathbb{R}}(qa - \tau(q)) \tag{3.14}$$

for all a.

For the existence of the limit (3.12) see remark 3.11.

Proof

The theorem of Gärtner-Ellis [27, Thm II] allows to estimate the exponential decay rate of the probabilities $P_n[s_K^{(n)} \in E]$ of our random variables $s_k^{(n)}$ being contained in a set E. Recall that the randomness is here in choosing the integer K from $\{0,\ldots,2^n-1\}$ with uniform distribution U_n. In the light of (3.11), the assumptions made in the theorem ensure that [27, Thm II] is applicable.

As is typical for results on large deviations, upper bounds are available for closed sets E while lower bounds can be obtained for open sets E. Recall that the range of allowed values for $s_k^{(n)}$ in $N^{(n)}(a,\varepsilon)$ is half open; thus

$$\#\{k : |s_k^{(n)} - a| < \varepsilon\} \le N^{(n)}(a,\varepsilon) \le \#\{k : |s_k^{(n)} - a| \le \varepsilon\}$$

Applying the LDP bounds of [27, Thm II] to the above sets gives immediately

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 N^{(n)}(a,\varepsilon) \le \sup_{|a'-a|\le\varepsilon} \tau^*(a')$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log_2 N^{(n)}(a,\varepsilon) \ge \sup_{|a'-a|<\varepsilon} \tau^*(a').$$

By continuity of $\tau^*(a)$ these two bounds on the right hand side coincide and (3.13) is established. Letting now $\varepsilon \to 0$ shows that $f(a) = \tau^*(a)$. ♢

The assumptions of this theorem are too restrictive in many applications. Before dwelling more on the relation between τ and f in section 4 let us note the following simple fact. Its proof explains why the Legendre transform appears in this context.

Lemma 3.6. *For any* $a \in \mathbb{R}$

$$f(a) \le \tau^*(a). \tag{3.15}$$

Proof

Fix $q \in \mathbb{R}$ and take a with $f(a) > -\infty$. Let $\gamma < f(a)$ and $\varepsilon > 0$. Then, there are arbitrarily large n such that $N^{(n)}(a,\varepsilon) \ge 2^{n\gamma}$. For such n we estimate $S^{(n)}(q)$ by noting

$$\sum_{k=0}^{2^n-1} 2^{-nqs_k^{(n)}} \ge \sum_{|s_k^{(n)}-a|<\varepsilon} 2^{-nqs_k^{(n)}} \ge N^{(n)}(a,\varepsilon) 2^{-n(qa+|q|\varepsilon)} \ge 2^{-n(qa-\gamma+|q|\varepsilon)} \tag{3.16}$$

and hence $\tau(q) \le qa - \gamma + |q|\varepsilon$. Letting $\varepsilon \to 0$ and $\gamma \to f(a)$, we find $\tau(q) \le qa - f(a)$. Since this is trivial if $f(a) = -\infty$ we have established that

$$\tau(q) \le qa - f(a) \quad \text{and} \quad f(a) \le qa - \tau(q) \quad \text{for all } a \text{ and } q \text{ in } \mathbb{R}. \tag{3.17}$$

From this it follows trivially that $\tau(q) \le f^*(q)$ and $f(a) \le \tau^*(a)$. ♢

Historical note In the special case when Y is the distribution function $\mathcal{M}$ of a measure μ, choosing the singularity exponent $s_k^{(n)} = \alpha_k^{(n)}$ results in

$$S_\alpha^{(n)}(q) = \sum_{k=0}^{2^n-1} \left|\mathcal{M}\left((k+1)2^{-n}\right) - \mathcal{M}(k2^{-n})\right|^q = \sum_{k=0}^{2^n-1} \left(\mu(I_k^{(n)})\right)^q. \tag{3.18}$$

This is the original form in which $\tau(q)$ as been introduced in multifractal analysis [45, 46, 36, 64]. Note that there is a close connection to the thermo-dynamical formalism [101].

3.4 Deterministic Envelopes

Often, we would like to use a simple analytical approach in order to gain intuition on the various spectra on a typical path of Y, or at least some estimate of them. To this end, we consider now position on the time axis, i.e., t or k_n, and the path of Y to be random simultaneously as we apply the LDP. More precisely, we consider the exponents $s_k^{(n)}(\omega)$ now as being random variables over $(\Omega \times 2^n)$ where k is picked with uniform distribution from $\{0, , \ldots, 2^n - 1\}$, and independently of ω.

Definition 3.7. *The* deterministic partition function *of Y is*

$$T(q) := \liminf_{n\to\infty} -\frac{1}{n} \log_2 \mathbb{E}_\Omega[S^{(n)}(q)]. \tag{3.19}$$

Remark 3.8. (Ergodic Processes) In the definitions of $\tau(q)$ and $T(q)$ we have assumed that Y is defined on a compact interval which we took to be $[0,1]$ without loss of generality. For processes defined on $\mathbb{R}$ we modify $S^{(n)}(q)$ to

$$S^{(n)}(q) := \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N2^n-1} 2^{-nqs_k^{(n)}}$$

and $N^{(n)}(a,\varepsilon)$ similarly. For *ergodic* processes this becomes $S^{(n)}(q) = 2^n \mathbb{E}_\Omega[2^{-nqs_k^{(n)}}]$ almost surely. Thus, $\mathbb{E}_\Omega[S^{(n)}(q)] = S^{(n)}(q)$ a.s. and

$$T(q) \stackrel{\text{a.s.}}{=} \tau(q,\omega). \tag{3.20}$$

We refer to (7.11) and (5.32) for an account on the extent to which marginal distributions may be reflected in multifractal spectra in general. ♣

For processes on $[0,1]$ we can not expect to have (3.20) in all generality. Nevertheless, (3.20) holds in various nice situations as we are going to see, and $T(q)$ does always serve as a *deterministic envelope* of $\tau(q,\omega)$:

Lemma 3.9. *For clarity, we make the randomness of τ explicit by writing $\tau(q,\omega)$. With probability one*

$$\tau(q,\omega) \geq T(q) \qquad \textit{for all } q \textit{ with } T(q) < \infty. \tag{3.21}$$

The inequality may be strict (see Example 5.3).

Proof

Consider any q with finite $T(q)$ and let $\varepsilon > 0$. Let n_0 be such that $\mathbb{E}_\Omega[S^{(n)}(q)] \leq 2^{-n(T(q)-\varepsilon)}$ for all $n \geq n_0$. Then,

$$\mathbb{E}\left[\limsup_{n\to\infty} 2^{n(T(q)-2\varepsilon)} S^{(n)}(q,\omega)\right] \leq \mathbb{E} \sum_{n\geq n_0} 2^{n(T(q)-2\varepsilon)} S^{(n)}(q,\omega) \leq \sum_{n\geq n_0} 2^{-n\varepsilon} < \infty.$$

Thus, almost surely $\limsup_{n\to\infty} 2^{n(T(q)-2\varepsilon)} S^{(n)}(q,\omega) < \infty$, and $\tau(q) \geq T(q) - 2\varepsilon$. It follows that this estimate holds with probability one simultaneously for all $\varepsilon = 1/m$ ($m \in \mathbb{N}$) and some countable, dense set of q values with $T(q) < \infty$. Since $\tau(q)$ and $T(q)$ are always concave due to corollary 4.3 below, they are continuous on open sets and the claim follows. ♢

Remark 3.10. (Importance of moments of negative order $q < 0$) In 'traditional' statistics moments are usually considered to be taken with respect to a centered random variable, i.e., $\mathbb{E}[|X - \mathbb{E}X|^q]$. In this setting, moments of negative order measure only the fluctuations around the mean and can, therefore, often be neglected.

As we pointed out in remark 3.8 $S^{(n)}(q)$ can be considered as approximating marginal moments of the random variables $2^{-ns_k^{(n)}}$, at least under some ergodicity assumption. Depending on one's choice these may be moduli of increments as in (3.18) or moduli of wavelet coefficients. Increments as well as wavelet coefficients are clearly centered for a process with zero mean increments and possess considerable mass around zero, especially for Gaussian processes such as fBm; in this case, negative order moments provide indeed only little information. We will comment in greater detail on the multifractal scaling of fBm and the rôle of negative q in Section 7.3 when the necessary results are available.

For processes with *positive increments* such as cascades, on the other hand, negative order moments become important and relevant, since they capture the probability of very small increments. In other words, the negative order moments are related to the time instances t with high regularity, i.e. the smooth parts in these otherwise 'spiky' processes.

A difficulty arises for cascades with fractal support. Due to boundary effects the coarse singularity exponents may become exceptionally large (yet finite), causing the partition function to degenerate for negative q [91]. In Section 5.6 we will show how to improve on the analysis using increment exponents $\alpha_k^{(n)}$ or Hölder exponents $h_k^{(n)}$. Similar problems are encountered also with wavelet exponents, where a remedy has been devised in [78] using local maxima in wavelet bands. ♣

Remark 3.11. (Quenched and annealed averages)

A simple application of Chebichev's inequality shows that $\mathbb{E}[\tau(q)] \geq T(q)$ which is clearly not as strong as lemma 3.9. However, $\mathbb{E}[\tau(q)]$ is of interest in itself. Assuming that the limit (3.12) actually exists, Dini's theorem allows to exchange expectation and limit and we may write

$$\mathbb{E}[\tau(q)] = \lim_{n\to\infty} (-1/n)\mathbb{E}[\log(S^{(n)}(q))].$$

In material science, this expression is also known as a *quenched average*. Exchanging expectation and logarithm –an operation which in general changes the object– we obtain $T(q)$, also termed *annealed average*.

The free energy is said to have the *self-averaging property* if quenched and annealed averages are equal. Since $\tau(q) \geq T(q)$ almost surely the free energy occurring naturally in the framework of multifractal analysis is self-averaging if and only if $\tau(q) = T(q)$ almost surely (provided the limits exist).

The existence of the limit (3.12) for binomial cascades has been established in [18] as well as in [38]. It follows also from the following simple observation, which promises wider applicability:

$$\limsup_{n\to\infty} -\frac{1}{n} \log_2 S^{(n)}(q) \leq \underline{f}^*(q).$$

This fact can be established similarly as lemma 3.6. Thus, the equality $\underline{f}(a) = T^*(a)$ entails the existence of the limit (3.12). ♣

The step from the partition function $\tau(q)$ to the deterministic envelope $T(q)$ consists in replacing averages of exponents within a path by averages within and across paths. For $f(a)$ this translates to replacing probability over one path by the probability within and across paths. As we shall see this means to average $N^{(n)}(a,\varepsilon)$ over all paths. For simplicity of the argument fix n, a and ε, and let $\mathbf{1}$ be the random variable which is 1 if $a-\varepsilon \le s_k^{(n)}(\omega) < a+\varepsilon$ and 0 otherwise, where (ω,k) are randomly chosen from $\Omega \times 2^n$. Obviously, $\mathbb{E}[\mathbf{1}] = P[\mathbf{1}=1]$. To compute this value we make use of Fubini's theorem and average $\mathbf{1}$ first within a fixed path –which yields $N^{(n)}(a,\varepsilon)/2^n$– and then average over all paths. Alternatively, we may fix the location k and average $\mathbf{1}$ over all paths first –which yields $P_\Omega[a-\varepsilon \le s_k^{(n)}(\omega) < a+\varepsilon]$– and then average over all k. In summary:

$$P_{\Omega\times 2^n}[a-\varepsilon \le s_k^{(n)} < a+\varepsilon] = \mathbb{E}_\Omega[N^{(n)}(a,\varepsilon)/2^n] = 2^{-n}\sum_{k=0}^{2^n-1} P_\Omega[a-\varepsilon \le s_k^{(n)} < a+\varepsilon]. \tag{3.22}$$

In analogy with (3.5) we multiply this probability with 2^n when defining the corresponding spectrum:

Definition 3.12. *The* deterministic grain spectrum *of Y is*

$$F(a) := \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\log_2 \mathbb{E}_\Omega[N^{(n)}(a,\varepsilon)] \tag{3.23}$$

To have some control over the convergence in n, which is needed to obtain a formula for f(a) valid for almost all paths in Section 8, we introduce

$$\underline{F}(a) := \lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log_2 \mathbb{E}_\Omega[N^{(n)}(a,\varepsilon)]. \tag{3.24}$$

Replacing $N^{(n)}(a,\varepsilon)$ by (3.22) in the proof of theorem 3.5 and taking expectations in (3.16) we find properties analogous to the pathwise spectra τ and f:

Theorem 3.13. *For all $a \in \mathbb{R}$*

$$F(a) \le T^*(a). \tag{3.25}$$

Furthermore, if $T(q)$ admits a finite limit as $n\to\infty$ for all $q\in\mathbb{R}$ similar to (3.12), and is concave and differentiable as a function of q, then $F(a)$ admits a limit as $n\to\infty$ analogous to (3.13), in particular

$$F(a) = \underline{F}(a) = T^*(a). \tag{3.26}$$

We give such an example in Example 5.3.

It follows from lemma 3.9 that with probability one $\tau^*(a,\omega) \le T^*(a)$ for all a. Similarly, the deterministic grain spectrum $F(a)$ is an upper bound to its pathwise defined random counterpart $f(a,\omega)$, however, only pointwise. On the other hand, we have here almost sure equality under certain conditions.

Theorem 3.14. *Fix some number $a \in \mathbb{R}$. Then, almost surely*

$$f(a,\omega) \leq F(a). \tag{3.27}$$

If for all n the random variables $s_k^{(n)}$ $(k = 0, \ldots, 2^n - 1)$ are i.i.d., and if $F(a) = \underline{F}(a) > 0$, then almost surely

$$f(a,\omega) = \underline{f}(a,\omega) = F(a). \tag{3.28}$$

Compare the regularity condition $F(a) = \underline{F}(a)$ (see (3.26)) to the more restrictive requirement that $F(a)$ assumes a limit similar to (3.13).

Remark 3.15. (Independent increments) It is easy to extend this result and allow $s_k^{(n)}$ to depend on some of its nearest neighbors, say on $s_l^{(n)}$ for $|l-k| < m_0$ for some constant m_0. Thus, if Y has independent increments, (3.28) applies not only to the increment exponents $\alpha_k^{(n)}$ but also to the Hölder exponents $h_k^{(n)}$ as well as to the wavelet exponents $w_k^{(n)}$ for compactly supported wavelets. ♣

Proof

The inequality (3.27) follows as in lemma 3.9, using the estimate

$$\mathbb{E}_\Omega \limsup_{n\to\infty} 2^{-n(F(a)+2\varepsilon)} N^{(n)}(a,\varepsilon) \leq \mathbb{E}_\Omega \sum_{n\geq m_0} 2^{-n(F(a)+2\varepsilon)} N^{(n)}(a,\varepsilon) \leq \sum_{n\geq m_0} 2^{-n\varepsilon}.$$

Since the grain spectra are not necessarily continuous, the inequality cannot be established for all a simultaneously, but only for a countable set of a-values.

Assume now that the $s_k^{(n)}$ $(k = 0, \ldots, 2^n - 1)$ are independent and identically distributed. To show equality in (3.27), let us note first that $N^{(n)}(a,\varepsilon)$ is a Bernoulli variable:

$$P_\Omega\left[N^{(n)}(a,\varepsilon) = j\right] = \binom{2^n}{j} {p_n}^j (1-p_n)^{2^n-j} \tag{3.29}$$

where $p_n = P_\Omega[a - \varepsilon \leq s_k^{(n)} < a+\varepsilon] = 2^{-n}\mathbb{E}_\Omega[N^{(n)}(a,\varepsilon)]$.

The property (3.26) says that the $P_\Omega[a-\varepsilon \leq s_k^{(n)} < a+\varepsilon]$ are 'close to' converging as $(n \to \infty)$. More precisely, (3.26) guarantees that for any $\rho > 0$ we find $\varepsilon_0(\rho)$ such that $\limsup_n$ and $\liminf_n$ of these quantities do not differ by more than ρ for all $\varepsilon < \varepsilon_0$. Thus, for any such ε and any $\eta > 0$ there is $n_1(\eta,\varepsilon,\rho)$ such that for all $n > n_1$

$$F(a) + \eta + \rho \geq \frac{1}{n}\log_2(2^n p_n) \geq F(a) - \eta - \rho. \tag{3.30}$$

Let now $\rho > 0$ and $\eta > 0$ be such that $1 > F(a) + \eta > F(a) - \eta > 0$. Using (3.30) it follows easily that $P_\Omega\left[N^{(n)}(a,\varepsilon) = j\right] / P_\Omega\left[N^{(n)}(a,\varepsilon) = j-1\right] > 1$, i.e., (3.29) grows monotonously as a function of j, for $j < 2^{nF(a)}$.

Now, let $0 < \gamma < F(a) - \eta - \rho$ and choose l such that $l - 1 < 2^{n\gamma} \leq l$. Then, exploiting the monotony of (3.29), one finds

$$P_\Omega\left[N^{(n)}(a,\varepsilon) \leq 2^{n\gamma}\right] \leq l \cdot \binom{2^n}{l} {p_n}^l (1-p_n)^{2^n-l}. \tag{3.31}$$

Using a standard estimate on binomial coefficients based on Stirling's formula and observing that $F(a) - \eta - \rho > 0$ a tedious but straightforward calculation allows to conclude that the RHS of (3.31) decays with hyper-exponential speed in n, mainly because of the last term. In summary,

$$\sum_{n=1}^{\infty} P_\Omega \left[N^{(n)}(a,\varepsilon) \le 2^{n\gamma} \right] < \infty.$$

By the Borel-Cantelli lemma

$$P_\Omega \left[N^{(n)}(a,\varepsilon) \le 2^{n\gamma} \text{ for infinitely many } n \right] = 0. \tag{3.32}$$

In other words, almost surely $N^{(n)}(a,\varepsilon) > 2^{n\gamma}$ for all large n, or again in other words, almost surely $\liminf_{n\to\infty}(1/n)\log_2 N^{(n)}(a,\varepsilon) \ge \gamma$. Letting $\varepsilon \to 0$ gives $\underline{f}(a,\omega) \ge \gamma$ a.s. Letting then $\gamma \to F(a) - \eta - \rho$, $\eta \to 0$ and $\rho \to 0$ (all in discrete sequences, of course) yields that almost surely

$$F(a) \le \underline{f}(a,\omega).$$

But $\underline{f}(a,\omega) \le f(a,\omega)$ by definition and $f(a,\omega) \le F(a)$ a.s. by (3.27). So, (3.28) follows. ♢

Here is a first result that allows to compute almost sure pathwise spectra from knowing only T under various regularity assumptions.

Corollary 3.16. *Assume that $T(q)$ admits a finite limit as $n \to \infty$ for all $q \in \mathbb{R}$, and is concave and differentiable as a function of q. Assume furthermore that for large n the singularity exponents $s_k^{(n)}$ ($k = 0, \dots, 2^n - 1$) used in T are i.i.d. random variables. Pick a such that $T^*(a) > 0$. Then almost surely*

$$\underline{f}(a,\omega) = f(a,\omega) = \tau^*(a,\omega) = F(a) = T^*(a). \tag{3.33}$$

Remark 3.17. (Negative Dimensions)

Note that T^* and F may assume negative values, which is not possible for f. Consequently, T^* and F may be expected to be a good estimator of f only where they are positive.

Negative $F(a)$ and $T^*(a)$ have been termed *negative dimensions* [67]. They correspond to probabilities of observing a coarse Hölder exponent a which decay faster than the 2^n 'samples' of $s_k^{(n)}$ available in one realization. Oversampling the process, i.e., analyzing several independent realizations will increase the number of samples and more 'rare' $s_k^{(n)}$ may be observed. In loose terms, in $\exp(-n\ln(2)F(a))$ independent traces one has a fair chance to see at least one $s_k^{(n)}$ of size $\simeq a$.

Thereby, it is essential not to average the spectra $f(a)$ of the various realizations but the numbers $N^{(n)}(a,\varepsilon)$. This way, negative 'dimensions' $f(a)$ become visible. ♣

4 The Multifractal Formalism

Various *multifractal spectra* have been introduced in the previous section, along with some simple relations between them which we may summarize as:

Corollary 4.1 (Multifractal formalism).
Provided all spectra are in terms of the same singularity exponent, we have

$$\dim(K^{[a]}) \leq \underline{f}(a) \leq f(a) \leq \tau^*(a) \overset{\text{a.s.}}{\leq} T^*(a). \tag{4.1}$$

The first relations hold pathwise, and the last one with probability one for every $a \in \mathbb{R}$. Similarly,

$$\dim(E^{[a]}) \leq f(a) \overset{\text{a.s.}}{\leq} F(a) \leq T^*(a).$$

The spectra on the left end have stronger implications on the local scaling structure while the ones on the right end are more easy to estimate or calculate.

This set of inequalities could fairly be called the 'multifractal formalism'. However, in the mathematical community a slightly different terminology is already established which goes as 'the multifractal formalism holds' and means that for almost all paths of a particular process the spectrum $\dim(K^{[a]})$ coincides with the Legendre transform of some adequate partition function (such as $\tau(q)$). It appears that this property holds indeed in a generic sense, meaning that in the proper context (see [52]) the 'multifractal formalism' is valid quasi-surely in Baire's sense. In view of (4.1) this implies that *equality* holds then between all introduced spectra.

Though we pointed out some conditions for equality between f, τ^* and T^* we must note that in general we may have strict inequality in some or all parts of (4.1). Such cases have been presented in [91] and [94]. There is, however, one equality which holds always and connects the two spectra in the center of (4.1).

Theorem 4.2. *Consider a realization or path of Y. Recall that infinite $s_k^{(n)}$ don't contribute to $\tau(q)$ nor to $f(a)$.*

a) Both-sided multifractal: *If the finite $s_k^{(n)}$ are bounded, then*

$$\tau(q) = f^*(q) \qquad \textit{for all } q \in \mathbb{R}. \tag{4.2}$$

b) Left-sided multifractal: *If the finite $s_k^{(n)}$ are unbounded from above but bounded from below, then*

$$\tau(q) = \begin{cases} f^*(q) & \textit{for all } q > 0 \\ -\infty & \textit{for all } q < 0. \end{cases}$$

c) Right-sided multifractal: *If the finite $s_k^{(n)}$ are bounded from above but unbounded from below, then*

$$\tau(q) = \begin{cases} -\infty & \textit{for all } q > 0 \\ f^*(q) & \textit{for all } q < 0. \end{cases}$$

d) *If the finite $s_k^{(n)}$ are unbounded from above and from below, then*

$$\tau(q) = -\infty \qquad \textit{for all } q \neq 0.$$

Proof
The following notation will be useful: $\kappa_n(a,\varepsilon) := \{k : a-\varepsilon \le s_k^{(n)} < a+\varepsilon\}$. Recall that $\tau(q) \le f^*(q)$ from lemma 3.6.

Now, to estimate $\tau(q)$ from below, we will group the terms in $S^{(n)}(q)$ conveniently, i.e.,

$$S^{(n)}(q) \le \Big(\sum_{i=-\lfloor \overline{a}/\varepsilon \rfloor}^{\lfloor \overline{a}/\varepsilon \rfloor} \sum_{\kappa_n(i\varepsilon,\varepsilon)} + \sum_{|s_k^{(n)}|>\overline{a}} \Big) 2^{-nqs_k^{(n)}}, \tag{4.3}$$

where we keep the choice of $\overline{a}$ open for the moment. Since we need uniform estimates on $N^{(n)}(a,\varepsilon)$ for various a, some preparation is needed.

Fix $q \in \mathbb{R}$ and let $\eta > 0$. Then, for every $a \in [-\overline{a}, \overline{a}]$ there is $\varepsilon_0(a)$ and $n_0(a)$ such that $N^{(n)}(a,\varepsilon) \le 2^{n(f(a)+\eta)}$ for all $\varepsilon < \varepsilon_0(a)$ and all $n > n_0(a)$. We would like to have ε_0 and n_0 independent from a for our uniform estimate. To this end note that $N^{(n)}(a',\varepsilon') \le N^{(n)}(a,\varepsilon)$ for all $a' \in [a-\varepsilon/2, a+\varepsilon/2]$ and all $\varepsilon' < \varepsilon/2$. By compactness we may choose a finite set of a_j $(j = 1,\ldots,m)$ such that the collection $[a_j - \varepsilon_0(a_j)/2, a_j + \varepsilon_0(a_j)/2]$ covers $[-\overline{a}, \overline{a}]$. Set $\varepsilon_1 = (1/2)\min_{j=1,\ldots,m} \varepsilon_0(a_j)$ and $n_1 = \max_{j=1,\ldots,m} n_0(a_j)$. Then, for all $\varepsilon < \varepsilon_1$ and $n > n_1$ we have

$$N^{(n)}(a,\varepsilon) \le 2^{n(f(a)+\eta)}$$

for all $a \in [-\overline{a}, \overline{a}]$. For $\varepsilon < \varepsilon_1$ and $n > n_1$ we estimate the first term in (4.3) by

$$\sum_{i=-\lfloor \overline{a}/\varepsilon \rfloor}^{\lfloor \overline{a}/\varepsilon \rfloor} \sum_{\kappa_n(i\varepsilon,\varepsilon)} 2^{-nqs_k^{(n)}} \le \sum_{i=-\lfloor \overline{a}/\varepsilon \rfloor}^{\lfloor \overline{a}/\varepsilon \rfloor} N^{(n)}(i\varepsilon,\varepsilon) 2^{-n(qi\varepsilon-|q|\varepsilon)} \le$$
$$\sum_{i=-\lfloor \overline{a}/\varepsilon \rfloor}^{\lfloor \overline{a}/\varepsilon \rfloor} 2^{-n(qi\varepsilon - f(i\varepsilon)-\eta-|q|\varepsilon)} \le (2\lfloor \overline{a}/\varepsilon \rfloor + 1)\cdot 2^{-n(f^*(q)-\eta-|q|\varepsilon)}. \tag{4.4}$$

Case a): For bounded $s_k^{(n)}$ this is all we need. Indeed, choosing $\overline{a}$ larger than $|s_k^{(n)}|$ for all n and k, the second term in (4.3) vanishes and (4.4) estimates $S^{(n)}(q)$ itself. Letting $n \to \infty$ we find $\tau(q) \ge f^*(q) - \eta - |q|\varepsilon$ for all $\varepsilon < \varepsilon_1$. Now we let $\varepsilon \to 0$ and finally $\eta \to 0$ to find the desired inequality $\tau(q) \ge f^*(q)$.

Case b): If the $s_k^{(n)}$ are bounded only from below we proceed differently for positive and negative q. For $q > 0$ we choose $\overline{a}$ large enough to ensure $q\overline{a} > f^*(q) + 1$ as well as $s_k^{(n)} > -\overline{a}$ for all k and n. The second term in (4.3) is then bounded by

$$\sum_{s_k^{(n)} > \overline{a}} 2^{-nqs_k^{(n)}} \le 2^n 2^{-nq\overline{a}} \le 2^{-nf^*(q)}$$

using that the sum has at the most 2^n terms. This expression is certainly smaller than the right hand side of (4.4), whence $S^{(n)}(q)$ is bounded by twice (4.4) and the result follows as before in case a).

For $q < 0$ we simply note that for any number x we can find arbitrarily large n such that $s_k^{(n)} > x$ for some k. This implies that $S^{(n)}(q) \geq 2^{-nqx}$ and $\tau(q) \leq qx$. Letting $x \to \infty$ proves the claim $\tau(q) = -\infty$ since q is negative.

Case c): If the $s_k^{(n)}$ are bounded only from above we argue similarly as in case b). Finally, in case d) we find $\tau(q) = -\infty$ for all $q \neq 0$ as in case b) and c). ♢

Note that $\tau(q)$ may be discontinuous at 0 in the one-sided cases, but as a Legendre transform[§] it is always concave. Moreover:

Corollary 4.3 (Properties of the partition function). *The partition function $\tau(q)$ is always concave. Depending on which case of theorem 4.2 applies, $\tau(q)$ is moreover continuous on $\mathbb{R}$, $\{q > 0\}$, or $\{q < 0\}$; furthermore, it is differentiable in those sets with at the most countable many exceptions.*

If for large enough n all $s_k^{(n)}$ are positive for a given path of the process Y, then $\tau(q)$ is *non-decreasing*. Such is the case for $s_k^{(n)} = \alpha_k^{(n)}$ and $s_k^{(n)} = h_k^{(n)}$ provided the path is increasing, or at least of bounded variation, and also for $s_k^{(n)} = w_k^{(n)}$ provided that the path lies in an appropriate regularity space such as L^2, the space of square-integrable functions. However, $\tau(q)$ may have decreasing parts if the analyzed process exists only in the distributional sense, such as binomial measures (see section 5 and Figure 7).

From a numerical point of view $\tau(q)$ is more robust than f since $\tau(q)$ involves averages and is not in terms of a double limit. Thus, one would like to invert theorem 4.2. In order to efficiently do so we need:

Lemma 4.4 (Lower semi-continuity of f and F). *Let a_m converge to a_*. Then*

$$f(a_*) \geq \limsup_{m\to\infty} f(a_m) \tag{4.5}$$

and verbatim for F.

Proof
For all $\varepsilon > 0$ one can find m_0 such that $a_* - \varepsilon < a_m - \varepsilon/2 < a_m + \varepsilon/2 < a_* + \varepsilon$ for all $m > m_0$. Then, $N^{(n)}(a_*, \varepsilon) \geq N^{(n)}(a_m, \varepsilon/2)$ and $\mathbb{E}[N^{(n)}(a_*, \varepsilon)] \geq \mathbb{E}[N^{(n)}(a_m, \varepsilon/2)]$. Following now only the case for f, we write

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 N^{(n)}(a_*, \varepsilon) \geq \limsup_{n\to\infty} \frac{1}{n} \log_2 N^{(n)}(a_m, \varepsilon/2) \geq f(a_m)$$

for any $m > m_0(\varepsilon)$. Letting first $m \to \infty$ and then $\varepsilon \to 0$ proves the claim. ♢

Corollary 4.5 (Central multifractal formalism). *We always have*

$$f(a) \leq f^{**}(a) = \tau^*(a). \tag{4.6}$$

For some a we have even equality. More precisely, for all $q \in \mathbb{R}$ for which $\tau'(q\pm)$ is meaningful

$$f(a) = \tau^*(a) = q\tau'(q\pm) - \tau(q\pm) \qquad \text{at } a = \tau'(q\pm). \tag{4.7}$$

[§]For a tutorial on the Legendre transform see [89, App. A].

Note that (4.7) uses only local properties of $\tau(q)$ which should be compared to theorem 3.5.

For a concave, real-valued function $g(q)$ the derivative g' exists in all but countable many exceptional points and it is a monotone function. Thus, one can define meaningfully $g'(q+)$ for *all* q as the limit of $g'(q_n)$ where q_n approaches q from the right and runs through any set of points where the derivative exists. The only difficulty with $\tau(q)$ lies then in its possibly infinite values. According to theorem 4.2 and corollary 4.3 $\tau'(q\pm)$ is meaningful either for all $q \in \mathbb{R}$, all $q \geq 0$, all $q \leq 0$, or none.

Remark 4.6. (Special values of τ and the shape of f) By convention, $S^{(n)}(0)$ counts how many of the $s_k^{(n)}$ are finite. So, $S^{(n)}(0) \geq 0$ and we have always $\tau(0) \leq 0$. Moreover, $S^{(n)}(0) \geq N^{(n)}(a, \varepsilon)$ for all a, which implies

$$f(a) \leq -\tau(0) \tag{4.8}$$

with equality at $a = \tau'(0\pm)$, lest the finite $s_k^{(n)}$ are unbounded from below and above. If all $s_k^{(n)}$ are finite, then $\tau(0) = -1$. In general, $-\tau(0)$ gives the (box) *dimension of the support* of the analyzed path of the process Y (compare Example 5.4 and [91]).

In some cases, such as $s_k^{(n)} = h_k^{(n)}$, the singularity exponents are monotonous in the sense that $2^{-ns_k^{(n)}} \geq \max(2^{-ns_{2k}^{n+1}}, 2^{-ns_{2k+1}^{n+1}})$. If so, $S^{(n)}(1) \geq 2S^{(n+1)}(1)$ and $\tau(1) \geq -1$ ($h_k^{(n)}$ are always bounded from below). In the 'classical case' of $s_k^{(n)} = \alpha_k^{(n)}$ for an *increasing* path of Y we have obviously $S^{(n)}(1) = S^{(n+1)}(1)$ and $\tau(1) = 0$. Consequently,

$$\text{Increasing path:} \qquad f_\alpha(a) \leq a \tag{4.9}$$

with equality at both, $a = \tau'(1+)$ and $a = \tau'(1-)$ (see Figure 6). It cannot be stressed enough that (4.9) may fail for processes which are not monotonous (compare Section 7.3). For certain increasing processes, $\tau'(1)$ has been identified as the *dimension of the carrier* and received the name 'information dimension' [44, 42, 43, 83, 32, 91, 6, 37] (compare (5.27)). One can think of the support as a function's 'realm', i.e the closure of all points where the function is non-constant, and of the carrier as the function's 'core', i.e., the smallest set of points one can choose over which the function still exhibits its full variation. For a cumulative distribution function of a probability measure, the support collects all but the open intervals with zero probability, while the core marks the smallest set with probability one.

♣

Proof

The graph of f^{**} is the concave hull of the graph of f which implies (4.6). It is an easy task to derive (4.7) under assumptions suitable to make the tools of calculus available such as continuous second derivatives. The reader not familiar with Legendre transforms should take the opportunity and perform a simple minimization of $qa - \tau(q)$ as a function of q to convince himself of (4.7).

To prove (4.7) in general let us first assume that τ is differentiable at a fixed q_*. Since $\tau(q_*) = f^*(q_*)$ we can find a sequence a_m such that $\tau(q_*) = \lim_{m\to\infty} q_* a_m - f(a_m)$.

Now consider the functions $q \mapsto qa_m - f(a_m)$. Due to (3.17) they must all lie above $\tau(\cdot)$. But evaluated at q_* these functions approach $\tau(q_*)$. Being linear, their slopes must converge to the slope of the tangent at $\tau(q)$ in q_*, in other words, a_m must converge to $a_* := \tau'(q_*)$.

From this, $f(a_m)$ converges to $q_*a_* - \tau(q_*)$. Applying lemma 4.4 we find that $f(a_*) \geq \lim_m f(a_m) = q_*a_* - \tau(q_*)$. Recalling (3.17) again implies the desired equality $f(a_*) = q_*a_* - \tau(q_*)$.

Let us now consider any $q \in \mathbb{R}$ for which $\tau'(q+)$ is meaningful. Then, there is a sequence of numbers q_m larger than q in which τ is differentiable and which converges down to q. Consequently, $\tau'(q+) = \lim_m \tau'(q_m)$. The formula (4.7) being established at all q_m lemma 4.4 applies now with $a_m = \tau'(q_m)$ and with $a_* = \tau'(q+)$ to yield $f(\tau'(q+)) = f(a_*) \geq \limsup_m f(a_m) = \lim_m q_m\tau'(q_m+) - \tau(q_m+) = q\tau'(q+) - \tau(q+)$. Again, (3.17) furnishes the opposite inequality. A similar argument applies to $\tau'(q-)$.

$\diamondsuit$

Corollary 4.7. *If $T(q)$ is finite for some positive q then the finite $s_k^{(n)}$ are bounded from below for almost all paths; moreover, for all $q > 0$ with finite $T(q)$*

$$T(q) = F^*(q). \tag{4.10}$$

If $T(q)$ is finite for some negative q then the finite $s_k^{(n)}$ are bounded from above for almost all paths, and (4.10) holds for all $q < 0$ with finite $T(q)$. Moreover, for all q for which $T'(q)$ exists

$$F(a) = T^*(a) = qT'(q) - T(q) \qquad \text{at } a = T'(q). \tag{4.11}$$

See Example 5.3 for a case with $F(a) < T^*(a)$.

Proof

If $T(q)$ is finite then $\tau(q) > -\infty$ by lemma 3.9. So, the first claims follow from theorem 4.2 and by taking expectation in (4.3) and (4.4). Formula (4.11) could actually be strengthened to read similar to (4.7), the proof following along the lines of corollary 4.5.

$\diamondsuit$

5 Binomial Multifractals

The binomial cascade has a long standing tradition in serving as the paradigm of multifractal scaling [64, 56, 66, 15, 49, 10, 91, 95]. We present it here with an eye on possible generalizations of use in modelling increasing random processes.

5.1 Construction of conservative cascades

The binomial cascades form a class of *increasing* processes. The construction of a binomial cascade relies on two main ingredients: the first one provides infinite geometric detail through iterative sub-division while the second one provides randomness. To clearly distinguish these two parts we construct a binomial cascade pathwise, i.e., we first describe the geometrical detail inherent to any path (or realization) of the cascade and introduce randomness only afterwards.

Each path of a binomial cascade is a *distribution* function $\mathcal{M}_b(t)$, in particular each path is right-continuous and increasing. As we will see, almost all paths possess no derivative (or density function), meaning that $\mathcal{M}_b(t) \neq \int_{-\infty}^{t} \mathcal{M}_b'(s)\mathrm{d}s$; unless for a handful of trivial cascades. Being distribution functions, though, each path $\mathcal{M}_b$ is uniquely related to a measure or (probability) distribution μ_b through $\mathcal{M}_b(t) = \mu_b(]-\infty, t[)$. Actually, the easiest way to construct $\mathcal{M}_b$ is by defining μ_b first. Except for trivial cases, μ_b is a true distribution and not a function in the usual sense. We will call μ_b a *binomial measure* and $\mathcal{M}_b(t) = \mu_b(]-\infty, t[)$ a *binomial cascade*.

In order to define μ_b we make the notation (2.4) more precise in the following. For any fixed t there is a unique sequence $k_1, k_2, \ldots$ such that the dyadic intervals $I_{k_n}^{(n)} = [k_n 2^{-n}, (k_n+1)2^{-n}[$ contain t for all integer n, indeed, $k_n = k_n(t) = \lfloor t2^n \rfloor$. It is clear that any t defines in this way a unique sequence k_n ($n \in \mathbb{N}$) with the following property:

Definition 5.1. *We call a sequence* $(k_1, k_2, \ldots)$ *such that* $I_{k_{n+1}}^{(n+1)}$ *is a subinterval of* $I_{k_n}^{(n)}$ *a* nested sequence*.*

Vice versa, for an infinite nested sequence, the $I_k^{(n)}$ form a decreasing sequence of half open intervals which shrink down to a unique singleton $\{t\}$.

The idea behind a binomial measure is to redistribute the mass lying in $I_{k_n}^{(n)}$ among its two dyadic subintervals $I_{2k_n}^{(n+1)}$ and $I_{2k_n+1}^{(n+1)}$ in the proportions of certain given numbers $M_{2k_n}^{(n+1)}$ and $M_{2k_n+1}^{(n+1)}$. For consistency we require $M_{2k_n}^{(n+1)} + M_{2k_n+1}^{(n+1)} = 1$. To make this procedure meaningful and amendable to analysis, further restrictions will be imposed on the *multipliers* $M_k^{(n)}$ in an instant (see (i)-(iii) below).

Definition 5.2. *For given* multipliers $M_k^{(n)}$ *the* binomial measure *is defined by setting*

$$\mu_b(I_{k_n}^{(n)}) = M_{k_n}^{(n)} \cdot M_{k_{n-1}}^{(n-1)} \cdots M_{k_1}^{(1)} \cdot M_0^{(0)}, \tag{5.1}$$

and the binomial cascade *is given by setting* $\mathcal{M}_b(0) = 0$ *and*

$$\mathcal{M}_b((k_n+1)2^{-n}) - \mathcal{M}_b(k_n 2^{-n}) = \mu_b(I_{k_n}^{(n)}). \tag{5.2}$$

It is enough to define the mass (or probability) of dyadic intervals since any interval $]-\infty, t[$ can be written as a disjoint union of dyadic intervals $J^{(n)}$ and $\mathcal{M}_b(t) = \mu_b(]-\infty, t[= \sum_n \mu_b(J^{(n)})$. In particular, integrals (expectations) with respect to μ_b can

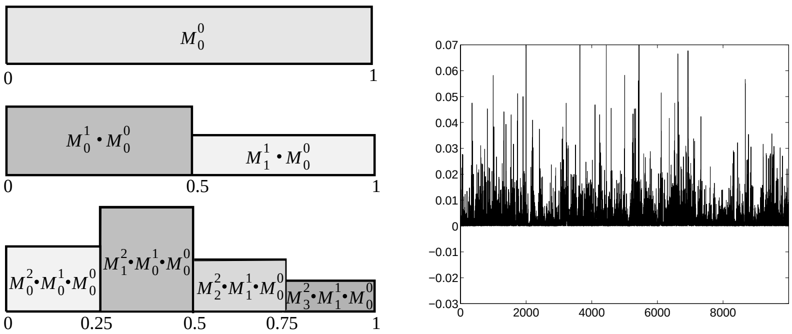

FIGURE 5. *Iterative construction of the binomial cascade.*

be calculated as

$$\int g(t)\mu_b(\mathrm{d}t) = \lim_{n\to\infty}\sum_{k=0}^{2^n-1} g(k2^{-n})\mu_b(I_k^{(n)}) \tag{5.3}$$

$$= \lim_{n\to\infty}\sum_{k=0}^{2^n-1} g(k2^{-n})\left(\mathcal{M}_b((k+1)2^{-n}) - \mathcal{M}_b(k2^{-n})\right)$$

$$= \int g(t)\mathrm{d}\mathcal{M}_b(t) \tag{5.4}$$

To say the same in other words, (5.2) defines $\mathcal{M}_b$ in all dyadic points. As a distribution function, $\mathcal{M}_b$ is continuous from the right side. Thus, knowing the function $\mathcal{M}_b$ in dyadic points is sufficient. Note that $\mathcal{M}_b$ is continuous at a given point t unless $M_{k_n(t)}^{(n)} = 1$ for all n large.

To generate randomness in $\mathcal{M}_b$, we choose the various $M_k^{(n)}$ to be random variables. The above properties hold then pathwise. We will make the following assumptions on the distributions of the multipliers $M_k^{(n)}$:

(o): Total mass. Any suitable, positive random variable may be chosen as $M_0^{(0)}$.

(i): Conservation of mass. Almost surely $M_k^{(n)}$ is positive and

$$M_{2k_n}^{(n+1)} + M_{2k_n+1}^{(n+1)} = 1 \tag{5.5}$$

for all n and k. As we have seen, this guarantees that $\mathcal{M}_b$ is well defined.

(ii): Nested independence. All multipliers of a nested sequence are mutually independent. As a consequence, for any nested sequence as in (5.2) we have

$$\mathbb{E}_\Omega[M_{k_n}^{(n)}\cdots M_0^{(0)}] = \mathbb{E}_\Omega[M_{k_n}^{(n)}]\cdots\mathbb{E}_\Omega[M_0^{(0)}] \tag{5.6}$$

and similar for other moments. This will allow for simple calculations in the sequel.

(iii): Identical distributions For all n and k

$$M_k^{(n)} \stackrel{\text{fd}}{=} \begin{cases} M_0 & \text{if } k \text{ is even,} \\ M_1 & \text{if } k \text{ is odd.} \end{cases} \tag{5.7}$$

A more general version of (iii) was given in [89] to allow for more flexibility in model fitting (see Example 7.2).

5.2 Relaxed conservation

The theory of cascades or, more properly, T-martingales [56, 10, 49, 8], provides a wealth of possible generalizations. Most importantly, it allows to remove the dependence between sibling multipliers enforced by the almost sure conservation condition (5.5) and to require only that

(i'): Conservation in the mean

$$\mathbb{E}_\Omega[M_0 + M_1] = 1. \tag{5.8}$$

Relaxing (i) into (i') brings about various peculiarities. Some of these, such as *virtual scaling exponents* require an appropriate context to be fully appreciated and we defer their discussion to Example 5.3. Here, we relate merely the changes in the construction that are required. Indeed, since the total increment is not preserved under iteration the definition of the cascade (5.1) needs to be adapted to read as:

$$\mu_b(I_{k_n}^{(n)}) = M_{k_n}^{(n)} \cdot M_{k_{n-1}}^{(n-1)} \cdots M_{k_1}^{(1)} \cdot M_0^{(0)} \cdot \lim_{m\to\infty} \sum_{i_m} M_{2k_n+i_1}^{(n+1)} \cdot M_{4k_n+i_2}^{(n+2)} \cdots M_{2^m k_n+i_m}^{(n+m)}. \tag{5.9}$$

Here, the sum runs over all nested sequences $(i_1, \dots, i_m)$ of length m. This sum reduces to 1 in the case of strict conservation (5.5), which follows by induction in m. Similarly, the expectation of the inner sum equals 1 due to (i'). Moreover, the sum forms a positive martingale and as such converges. This makes (5.9) meaningful.

The limit, however, may *degenerate* and vanish almost surely. This may happen despite the conservation of mass in the mean, because products of multipliers might be very small with quite large probability and the binomial cascades 'dies out'. This case is equivalent with $\mathbb{E}[\mathcal{M}_b(1)] = 0$, since $\mathcal{M}_b(1)$ is a positive random variable, as well as equivalent with $T'(1) \le 0$ (see [56]). For an intuitive reasoning we mention that $T'(1)$ is the almost sure dimension of the carrier of the cascade whenever it is positive (see [56], remark 5.8 and Example 5.3). Let us assume for the remainder that $T'(1) > 0$; then $\mathbb{E}[\mathcal{M}_b(1)] = \mathbb{E}[M_0^{(0)}]$.

We noted that $\mathcal{M}_b(1)$ is in general not equal to $M_0^{(0)}$. Similarly, $\mu_b(I_k^{(n)})$ is not equal to $M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot M_0^{(0)}$ in general. The 'identically distributed multipliers' (ii), however, imply that $\mu_b(I_{k_n}^{(n)})$ is distributed as

$$\mu_b(I_{k_n}^{(n)}) \stackrel{\text{d}}{=} M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot \mathcal{M}_b(1). \tag{5.10}$$

Mandelbrot [66] calls $M_{k_n}^{(n)} \cdots M_{k_1}^{(1)}$ the *low frequency* part of $\mu_b(I_{k_n}^{(n)})$, and the subtree of multipliers 'hanging off' below the node k_n –which essentially constitute the limiting part distributed as $\mathcal{M}_b(1)$– the *high frequency* part. Since $\mathbb{E}[\mathcal{M}_b(1)] = \mathbb{E}[M_0^{(0)}]$ we get

$$\mathbb{E}[\mu_b(I_{k_n}^{(n)}) \mid M_{k_n}^{(n)}, \ldots, M_{k_1}^{(1)}] = M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot \mathbb{E}[M_0^{(0)}]. \tag{5.11}$$

and $\mathbb{E}[\mu_b(I_{k_n}^{(n)})] = 1/2^n \mathbb{E}[M_0^{(0)}]$.

For *numerical simulation* (5.11) reminds us that an iterative computation of the products $M_{k_n}^{(n)} \cdots M_{k_1}^{(1)}$ does not provide the *value* of the actual process increment $\mu_b(I_k^{(n)})$ –as is the case with strict conservation (5.5)– but rather what its value could be *expected* to be if the construction continued.

The main advantage of relaxing (5.5) is that we can use unbounded multipliers M_0 and M_1 such as log-normal random variables. In this particular case, the marginals of the increment process, i.e., $\mu_b(I_k^{(n)})$ are exactly log-normal on all scales. For general binomials with (i'), (ii) and (iii) it can be argued that the marginals $\mu_b(I_k^{(n)})$ are at least *asymptotically log-normal* by applying the Central Limit Theorem provided the logarithm of (5.2) is of finite variance.

5.3 Wavelet Decomposition

The scaling coefficients of μ_b using the Haar wavelet are simply

$$\text{Haar:} \qquad D_{n,k}(\mu_b) = \int \phi_{j,k}^*(t)\, \mu_b(\mathrm{d}t) = 2^{n/2} \int_{k2^{-n}}^{(k+1)2^{-n}} \mu_b(\mathrm{d}t) = 2^{n/2} \mu_b(I_k^{(n)}) \tag{5.12}$$

from (2.8) and (5.3). With (2.9) and (5.2) we get the explicit expression for the Haar wavelet coefficients:

$$\text{Haar:} \qquad 2^{-n/2} C_{n,k_n}(\mu_b) = \mu_b(I_{2k_n}^{(n+1)}) - \mu_b(I_{2k_n+1}^{(n+1)}) = \prod_{i=0}^{n} M_{k_i}^{(i)} (M_{2k_n}^{(n+1)} - M_{2k_n+1}^{(n+1)}). \tag{5.13}$$

To obtain formulas for general wavelets let us start with the observation that for any *function* ψ supported on $[0,1]$

$$2^{-n/2} C_{n,k_n}(\mu_b) = \int_{I_{k_n}^{(n)}} \psi(2^{(n)} t - k_n) \mu_b(\mathrm{d}t) = M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot \int_0^1 \psi(t') \mu_b^{(n,k_n)}(\mathrm{d}t'). \tag{5.14}$$

This comes about from (5.2) and (5.3). Here $\mu_b^{(n,k_n)}$ is the binomial measure constructed from the subtree which has as its root at the node k_n of level n of the original tree:

Definition 5.3. *A* binomial sub-cascade *is defined for any given nested sequence* $k_1, \ldots, k_n$ *by setting*

$$\mu_b^{(n,k_n)}(I_{i_m}^{(m)}) = \tilde{M}_0^{(0)} M_{2k_n+i_1}^{(n+1)} \cdot M_{4k_n+i_2}^{(n+2)} \cdots M_{2^m k_n + i_m}^{(n+m)} \tag{5.15}$$

for any nested sequence $(i_1, \ldots, i_m)$*, where* $\tilde{M}_0^{(0)}$ *is an independent copy of* $M_0^{(0)}$*. Also, we set* $\mathcal{M}_b^{(n,k_n)}(t) := \mu_b^{(n,k_n)}([0,t])$*. The generalization to cascades with conservation in the mean is obvious, but tedious.*

With little more effort we compute the analogue to (5.14) for $\mathcal{M}_b$ itself. In preparation let us note that for $k_n = k_n(t) = \lfloor t2^n \rfloor$

$$\mathcal{M}_b(t) - \mathcal{M}_b(k_n 2^{-n}) = \mu_b([k_n 2^{-n}, t]) = M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \mathcal{M}_b^{(n,k_n)}(2^n t - k_n). \tag{5.16}$$

Let us assume now that $\int \psi = 0$ and that the function ψ is supported on $[0,1]$, whence $\psi 2^n \cdot -k)$ is supported on $I_k^{(n)}$. Substituting $t' = 2^n t - k_n$ yields

$$\begin{aligned}\int \psi(2^n t - k_n)\mathcal{M}_b(t)\mathrm{d}t &= \int_{I_{k_n}^{(n)}} \psi(2^n t - k_n)\left(\mathcal{M}_b(t) - \mathcal{M}_b(k_n 2^{-n})\right)\mathrm{d}t \\ &= 2^{-n} \cdot M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot \int_0^1 \psi(t')\mathcal{M}_b^{(n,k_n)}(t')\mathrm{d}t'. \end{aligned} \tag{5.17}$$

For convenience let us introduce

$$A_{n,k}(\mu_b) := \int_0^1 \psi(t)\mu_b^{(n,k)}(\mathrm{d}t) \qquad \text{and} \qquad A_{n,k}(\mathcal{M}_b) := \int_0^1 \psi(t)\mathcal{M}_b^{(n,k)}(t)\mathrm{d}t. \tag{5.18}$$

Lemma 5.4. *Let ψ be a wavelet supported on $[0,1]$, and assume that the binomial measure μ_b satisfies (i)-(iii) (see (5.5)). Then,*

$$C_{n,k_n}(\mu_b) = 2^{n/2} \cdot M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot A_{n,k_n}(\mu_b), \tag{5.19}$$

and if ψ is admissible ($\int \psi = 0$)

$$C_{n,k_n}(\mathcal{M}_b) = 2^{-n/2} \cdot M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot A_{n,k_n}(\mathcal{M}_b). \tag{5.20}$$

Moreover, for both, μ_b and $\mathcal{M}_b$ the random variables A_{n,k_n} and $M_{k_i}^{(i)}$ ($i = 1, \ldots, n$) are mutually independent, and

$$A_{n,k_n} \stackrel{\mathrm{d}}{=} A_{0,0} = C_{0,0} \tag{5.21}$$

Proof
Use (2.8), (5.14) and (5.17) to obtain the formulas. Independence follows from the nested independence (5.6). Finally, the identical distributions of the multipliers (5.7) imply that $\int_0^1 \psi(t)\mu_b^{(n,k_n)}(\mathrm{d}t)$ is equal in distribution to $C_{0,0} = \int_0^1 \psi(t)\mu_b(\mathrm{d}t)$. ♢

As a particular example we find for the Haar wavelet (use (5.13))

$$A_{n,k_n} = M_{2k_n}^{(n+1)} - M_{2k_n+1}^{(n+1)} \stackrel{\mathrm{d}}{=} M_0 - M_1 = C_{0,0}.$$

The deterministic version has also been observed in [7].

It is obvious that the dyadic structure present in both, the construction of the binomial measure as well as in the wavelet transform, are responsible for the simplicity of the computation above. It is, however, standard by now to extend the procedure to more general multi-nomial cascades such as $\mathcal{M}_c$ introduced in Section 5.6 (see [6, 91]).

5.4 Multifractal Analysis of the Binomial Cascade

Almost all of the 'classical' scaling exponents $h_k^{(n)}$, $\alpha_k^{(n)}$ and $w_k^{(n)}$ can be dealt with explicitly for the cascades. Lemma 5.4 provides $w_k^{(n)}$ readily for both, $\mathcal{M}_b$ and μ_b. Through (5.2) one easily finds $\alpha_k^{(n)}$ and eventually $h_k^{(n)}$ for $\mathcal{M}_b$; but increments as they appear in $\alpha_k^{(n)}$ and $h_k^{(n)}$ are not well defined for the distribution μ_b.

For the sake of simplicity and to retrace history let us start with the well known multifractal analysis of $\mathcal{M}_b$ based on increments. We will return to the wavelet analysis in Section 6.2. As a starter we establish the good news that for increasing processes the partition functions obtained using $\alpha_k^{(n)}$ or $h_k^{(n)}$ coincide.

Lemma 5.5. *Assume that $\mathcal{M}$ is almost surely increasing. Then, for all q*

$$T_\alpha(q) = T_h(q) \tag{5.22}$$

and for almost all paths

$$\tau_\alpha(q,\omega) = \tau_h(q,\omega). \tag{5.23}$$

Proof

To understand the proof note that the task of estimating partition sums against each other is two-fold: estimate the terms of the sums themselves, and ensure that each term of one sum has a counterpart in the other sum.

Consider an increasing path $\mathcal{M}(t)$. For notational simplicity we set

$$\Delta_k^{(n)} = \mathcal{M}\big((k+1)2^{-n}\big) - \mathcal{M}(k2^{-n}) \quad \text{and} \quad \tilde{\Delta}_k^{(n)} = \mathcal{M}\big((k+2)2^{-n}\big) - \mathcal{M}((k-1)2^{-n}).$$

Recall that we have $h_k^{(n)} = -(1/n)\log_2 \tilde{\Delta}_k^{(n)}$, while $\alpha_k^{(n)} = -(1/n)\log_2 \Delta_k^{(n)}$.

The easy part of the proof is to note that $\Delta_k^{(n)} \leq \tilde{\Delta}_k^{(n)}$ by monotony of $\mathcal{M}$. Since there is a one-to-one correspondence of terms we find, by summing over all k that $S_h^{(n)}(q) \leq S_\alpha^{(n)}(q)$ for $q<0$ and $S_\alpha^{(n)}(q) \leq S_h^{(n)}(q)$ for $q \geq 0$. Taking limits the estimates translate immediately into $\tau_h(q) \geq \tau_\alpha(q)$ for $q<0$ and $\tau_\alpha(q) \geq \tau_h(q)$ for $q \geq 0$. By taking first expectation and then limits we obtain inequalities between the envelopes $T(q)$.

To get inequalities in the other direction (and thus establish equality) we claim first that $S_\alpha^{(n)}(q) \leq S_h^{(n+2)}(q)$ for $q<0$. The difference of the indices n and $n+2$ will not matter in the limit. For each $I_k^{(n)}$ there is $m(k)$ such that

$$[(m(k)-1)2^{-n-2}, (m(k)+2)2^{-n-2}] \subset I_k^{(n)}, .$$

Obviously, $\tilde{\Delta}_{m(k)}^{(n+2)} \leq \Delta_k^{(n)}$ again by monotony. Since $m(k) \neq m(k')$ when $k \neq k'$ we get for $q<0$ that $S_\alpha^{(n)}(q) \leq \sum_k (\tilde{\Delta}_{m(k)}^{(n+2)})^q < S_h^{(n+2)}(q)$, as claimed.

The last inequality is strict since there are only 2^n indices of the form $m(k)$, while $S_h^{(n+2)}(q)$ runs over the entire range $m = 0, \ldots, 2^{n+2}-1$. For this reason, the above argument does not help for $q>0$. To obtain an estimate for $q>0$ we need to construct a 'counter-term' $\Delta_k^{(n)}$ for *each* $\tilde{\Delta}_m^{(n+2)}$. But each interval of the form $[(m-1)2^{-n-2}, (m+$

$2)2^{-n-2}]$ intersects exactly two $I_k^{(n)}$-s. Let us index the one with the larger increment by $k(m)$. We have then $\tilde\Delta_m^{(n+2)} \le 2\Delta_{k(m)}^{(n)}$ by monotony of $\mathcal{M}$. Clearly, $k(m)=k(m')$ is possible even for $m \neq m'$. But each $I_k^{(n)}$ intersects only 6 intervals of the form $[(m-1)2^{-n-2},(m+2)2^{-n-2}]$ and at the most 6 different values of m can lead to the same value $k(m)$. This yields $S_h^{(n+2)}(q) \le \sum_m (2\Delta_{k(m)}^{(n)})^q \le 6\cdot 2^q \cdot S_\alpha^{(n)}(q)$ for $q \ge 0$, completing the proof. ♢

After this preparation let us now compute the multifractal envelope of a binomial cascade for $\alpha_k^{(n)}$. Recall that $\mathcal{M}_b((k_n+1)2^{-n}) - \mathcal{M}_b(k_n) = \mu_b(I_{k_n}^{(n)})$. Using in sequence (5.1) and (5.10), the nested independence (ii) (see (5.6)) and identical distributions (5.7) we obtain

$$\begin{aligned}
\mathbb{E}[S^{(n)}_{\alpha,\mathcal{M}_b}(q)] &= \sum_{k_n=0}^{2^n-1} \mathbb{E}\Big[\Big(M_{k_n}^{(n)}\Big)^q \cdots \Big(M_{k_1}^{(1)}\Big)^q\Big]\ \mathbb{E}[(\mathcal{M}_b(1))^q] \\
&= \mathbb{E}[(\mathcal{M}_b(1))^q]\cdot \sum_{i=0}^{n} \binom{n}{i} \mathbb{E}\,[M_0^q]^i\, \mathbb{E}\,[M_1^q]^{n-i} \\
&= \mathbb{E}[(\mathcal{M}_b(1))^q]\cdot (\mathbb{E}[M_0^q] + \mathbb{E}[M_1^q])^n \,. \qquad (5.24)
\end{aligned}$$

Recall that $\mathcal{M}_b(1) = M_0^{(0)}$ almost surely if the strict conservation (5.5) holds.

Theorem 5.6. *Assume that $\mathcal{M}_b$ is a binomial cascade for which (i'), (ii) and (iii) hold (see (5.8)). Then, $T(q) = -\infty$ if either of $\mathbb{E}[M_0^q]$, $\mathbb{E}[M_1^q]$ or $\mathbb{E}[(\mathcal{M}_b(1))^q]$ is ∞. Otherwise,*

$$T_{\alpha,\mathcal{M}_b}(q) = T_{h,\mathcal{M}_b}(q) \;=\; -\log_2 \mathbb{E}[(M_0)^q + (M_1)^q]\,. \qquad (5.25)$$

Assume furthermore, that the multipliers M_0 and M_1 have at least some finite moment of negative order. Then, when using increment exponents $\alpha_k^{(n)}$, the following holds with probability one for all a such that $T^*(a) > 0$:

$$\dim(K^{[a]}) = \dim(E^{[a]}) = \underline{f}(a) = f(a) = \tau^*(a) = F(a) = T^*(a). \qquad (5.26)$$

Assume the stronger condition, that the multipliers M_0 and M_1 are actually bounded away from zero. Then, when using increment exponents $h_k^{(n)}$, for any a with $T^*(a) > 0$ the same equality (5.26) holds almost surely.

The theorem and (5.26) assert – in the first case– that almost all paths possess the same entire spectra, while in the second case the spectra can be guaranteed to be equal only in a fixed, countable set of a-s. When interested only in a finite set of a-s, however, this result is obviously strong enough.

Remark 5.7. (Wavelet analysis and fractal support) In essence the same formulas (5.25) and (5.26) determine the spectra emerging from an analysis of $\mathcal{M}_b$ through certain analyzing wavelet exponents $w_k^{(n)}$ (see Section 6.2).

There is a natural extension of this theorem to multi-nomial cascades; however, for multi-nomial cascades with *fractal* support the scaling exponents $h_k^{(n)}$ and $\alpha_k^{(n)}$ have to be adapted [91] to yield meaningful partition functions (see Section 5.6 and (5.38)). ♣

Remark 5.8. (Degenerate cascades) Recall that binomial cascades with conservation in the mean (see i') degenerate, i.e., $\mathbb{E}[\mathcal{M}_b(1)] = 0$, exactly when $T'_\alpha(1) \leq 0$ (see [56]). Since

$$T'(1) = -\mathbb{E}[M_0 \log_2(M_0) + M_1 \log_2(M_1)] \tag{5.27}$$

this condition means that 'on the carrier' we multiply mass per iteration step 'in the average' with a number smaller than 1, and so the cascade 'dies out'. The expression (5.27) for $T'(1)$ justifies the name 'information dimension' (compare remark 4.6 and Section 5.2). ♣

Proof
Formula (5.25) is already established. Corollary 4.7 shows that $F(a) = T^*(a)$. The hard part of the theorem is, of course, to find a formula for $\dim(K^{[a]})$. Assuming finite moments of negative order for the multipliers $\dim(K^{[a]})$ using the scaling exponents $\alpha_k^{(n)}$ was computed in [8]. It is easy to verify that the formula of [8] for $\dim(K^{[a]})$ coincides with $T^*(a)$ as given in (5.25). Earlier results such as [33] (on $\alpha_k^{(n)}$) and [6] (on $h_k^{(n)}$) obtained the same formula. These papers use the more restrictive assumption that the multipliers are bounded away from zero, but they are somewhat easier to read. Corollary 4.1 implies (5.26) and the proof is complete. ♢

5.5 Examples

Example 5.1. (βeta Binomial) In our first example we choose the multipliers M_0 and M_1 to follow a β distribution, which has the density $c_p t^{p-1}(1-t)^{p-1}$ for $t \in [0,1]$ and 0 elsewhere. Hereby, $p > 0$ is a parameter and c_p is a normalization constant. Note that the conservation of mass (5.5) imposes a symmetrical distribution once we decide to have M_0 and $M_1 = 1 - M_0$ equally distributed.

The β distribution has finite moments of order $q > -p$ which can be expressed explicitly using the Γ-function. We get

$$\beta\text{-Binomial:} \qquad T_\alpha(q) = -1 - \log_2 \frac{\Gamma(p+q)\Gamma(2p)}{\Gamma(2p+q)\Gamma(p)} \qquad (q > -p), \tag{5.28}$$

and $T(q) = -\infty$ for $q \leq -p$. For a typical shape of these spectra see Figure 6.

In [89] binomial cascades were successfully used for modelling data traffic on the internet. To provide more flexibility the distributions of the multipliers were allowed to depend on the scale n (compare Example 7.2). ♠

Example 5.2. (Uniform Binomial) As a special case of the βeta binomial we obtain uniform distributions for the multipliers when setting $p = 1$. The formula (5.28) simplifies to $T_\alpha(q) = -1 + \log_2(1+q)$ for $q > -1$. Applying the formula for the Legendre transform (4.11) yields the explicit expression

$$\text{uniform binomial:} \qquad T^*_\alpha(a) = 1 - a + \log_2(e) + \log_2\left(\frac{a}{\log_2(e)}\right) \tag{5.29}$$

for $a > 0$ and $T^*_\alpha(a) = -\infty$ for $a \leq 0$. ♠

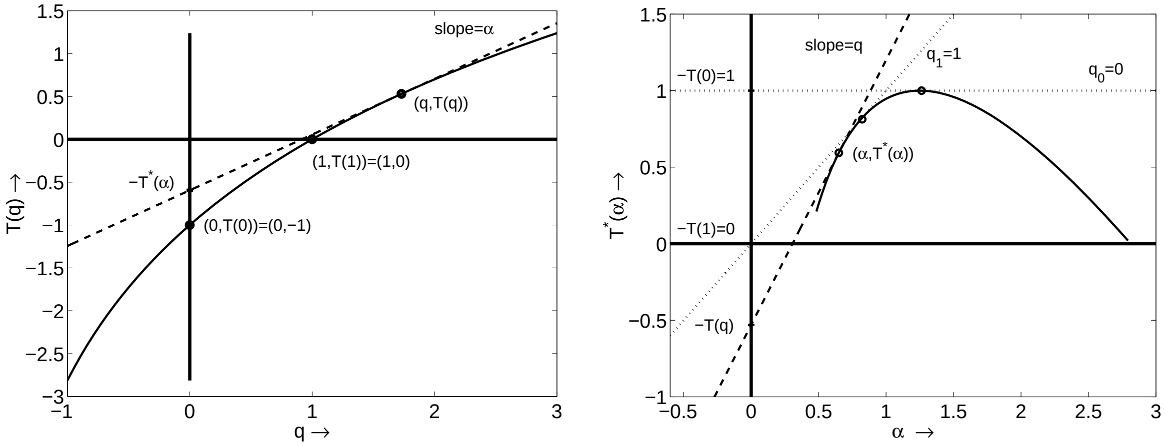

FIGURE 6. *The spectrum of a binomial measure with β distributed multipliers with* $p = 1.66$. *The formula is given by (5.28). Trivially,* $T(0) = -1$, *whence the maximum of* T^* *is* 1. *In addition, every positive increment process has* $T(1) = 0$, *whence* T^* *touches the bisector. Finally, the LRD parameter is* $H_{\text{var}} = (T(2) + 1)/2 = 0.85$*) (see (7.20) and (7.25) below).*

Example 5.3. (Log-normal Binomial) Another case of strong interest is the use of log-normal distributions for the multipliers M_0 and M_1. Note that we have to replace (i) by (i') (see (5.8)) in this case since log-normal variables can be arbitrarily large, i.e., larger than 1. We comment on some peculiarities this might cause after computing the spectrum.

Deterministic envelope: As the multipliers M_0 and M_1 of a log-normal binomial are log-normal variables we need expressions for the moments of such variables. Let G be a Gaussian with mean m and variance σ^2, and set $M = e^G$. Then, $\mathbb{E}[M^q] = \mathbb{E}[\exp(qG)] = \exp(qm + q^2\sigma^2/2)$.

For further simplification let us consider the case where M_0 and M_1 are equally distributed. Then, their mean must be $1/2$ which translates into $m + \sigma^2/2 = -\ln(2)$. We find from (5.25) that

$$\text{log-normal binomial:} \qquad T_\alpha(q) = (q-1)\left(1 - \frac{\sigma^2}{2\ln(2)}q\right) \tag{5.30}$$

for all $q \in \mathbb{R}$ such that $\mathbb{E}[(\mathcal{M}_b(1))^q]$ is finite. For later use note that the parabola in (5.30) has two zeros: 1 and $q_{\text{crit}} = 2\ln(2)/\sigma^2$.

Degenerate case and tail of marginals It is known also for more general cascades that $\mathbb{E}[\mathcal{M}_b(1)] = 0$ if and only if $T'(1) \leq 0$ [56]. Since $\mathcal{M}_b(1)$ is a positive random variable this occurs if and only if $\mathcal{M}_b(1) = 0$ almost surely, and we say that the cascade is *degenerate*.

To ensure non-degenerate cascades for the remainder we assume $T'(1) > 0$. Due to the parabolic shape of (5.30) this condition may be expressed easily as $q_{\text{crit}} > 1$, i.e., $\sigma^2 < 2\ln(2)$.

Assuming that $T'(1) > 0$, [56] shows that $T'(1)$ gives the almost sure dimension of the carrier of the cascade. Since paths with $\mathcal{M}_b(1) = 0$ have an empty carrier we conclude that actually $\mathcal{M}_b(1) \neq 0$ almost surely, not only with positive probability.

Always assuming $T'(1) > 0$, another result of [56] states that $\mathbb{E}[(\mathcal{M}_b(1))^q] < \infty$ exactly for $q < q_{\mathrm{crit}}$. Moreover, [66] conjectures and [75] proves the marginal tail behavior $P[\mathcal{M}_b(1) > x] \sim x^{-q_{\mathrm{crit}}}$. We will return with an explanation in an instant. For the moment we conclude that (5.30) is valid exactly for $q < q_{\mathrm{crit}}$, for all other q $T(q) = -\infty$.

Legendre spectrum: Since $T(q)$ is differentiable exactly for $q < q_{\mathrm{crit}}$ we may obtain its Legendre transform implicitly from (4.11) for $a = T'(q)$ with $q < q_{\mathrm{crit}}$, i.e., for all $a > a_{\mathrm{crit}} = T'(q_{\mathrm{crit}}) = \sigma^2/(2\ln(2)) - 1$. Eliminating q from (4.11) yields the explicit form

$$T^*_\alpha(a) = 1 - \frac{\ln(2)}{2\sigma^2}\left(a - 1 - \frac{\sigma^2}{2\ln(2)}\right)^2 \qquad (a \geq a_{\mathrm{crit}}). \tag{5.31}$$

For $a \leq a_{\mathrm{crit}}$ the Legendre transform yields $T^*(a) = a \cdot q_{\mathrm{crit}}$. Thus, at a_{crit} the spectrum T^* crosses over from the parabola (5.31) to its tangent through the origin with slope q_{crit} (the other tangent through the origin is the bisector).

Since $q_{\mathrm{crit}} > 1$ we have $a_{\mathrm{crit}} < 0$ and the spectrum possesses *negative* a with *finite*, though negative $T^*(a)$ (compare with the 'uniform' binomial cascade above which has no such a). We return to this fact in an instant. The two zeros of T^* are both positive for any choice of σ, since T^* is —apart from the linear part for $a < a_{\mathrm{crit}}$– a parabola which touches the bisector and has a maximum value of 1.

Rescaled histograms: The log-normal framework allows also to calculate $F(a)$ explicitly. We use the opportunity to demonstrate which rescaling properties of the marginal distributions of the increment processes of $\mathcal{M}_b$ are captured in the multifractal spectra.

First, note that $-\ln(2) \cdot \alpha_k^{(n)} = (1/n)\ln(\mu_b(I_k^{(n)}))$ (see (2.13) and (5.10)) is the sum of n independent Gaussian random variables $(1/n)\ln(M_k^{(n)})$ of mean m/n and variance σ^2/n^2, plus an independent additive term distributed as $(1/n)\ln(\mathcal{M}_b(1))$. To keep computations simple we neglect this last term; indeed, it will converge in distribution to 0 since $\ln(\mathcal{M}_b(1))$ has a Weibull tail and, thus, will not affect the distribution of $\alpha_k^{(n)}$ in the limit. In summary, this amounts to approximating $-\ln(2) \cdot \alpha_k^{(n)}$ by a Gaussian of mean m and variance σ^2/n. The mean value theorem of integration gives

$$\begin{aligned} P_\Omega[|\alpha_k^{(n)} - a| < \varepsilon] &\simeq \frac{1}{\sqrt{2\pi\sigma^2/n}} \int_{\ln(2)(-a-\varepsilon)}^{\ln(2)(-a+\varepsilon)} \exp\left(-\frac{(x-m)^2}{2\sigma^2/n}\right) \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \ln(2) \cdot 2\varepsilon \cdot \exp\left(-\frac{(-\ln(2)x_{a,n} - m)^2}{2\sigma^2/n}\right) \end{aligned}$$

with $x_{a,n} \in [a - \varepsilon, a + \varepsilon]$ for all n. To compute $F(a)$ we first need to sum over k (see (3.22)); however, the expression is identical for all k. Next, keeping only the exponential term in n and substituting $m = -\sigma^2/2 - \ln(2)$ we find for large n

$$\frac{1}{n}\log_2\left(2^n P_\Omega[|\alpha_k^{(n)} - a| < \varepsilon]\right) \simeq 1 - \frac{\ln(2)}{2\sigma^2}\left(x_{a,n} - 1 - \frac{\sigma^2}{2\ln(2)}\right)^2. \tag{5.32}$$

Taking finally the limit of $\varepsilon \to 0$ we see that $F(a)$ equals the right hand side of (5.31) for all a, and thus $T^*(a) = F(a)$ for all $a = T'(q)$ as stated in corollary 4.7. On the other hand, $T^*(a) > F(a)$ for $a < a_{\rm crit}$. The above computation shows impressively how well adapted a multiplicative iteration with log-normal multipliers is to the multifractal analysis (or vice versa): F extracts, basically, the exponent of the Gaussian kernel.

In the range of a with $F(a) > 0$ —where equality in the multifractal formalism is known to hold for $\mathcal{M}_b$ (see (5.26))— these features can be measured or estimated from a 'typical' realization via the *re-normalized histogram*, i.e., the grain based multifractal spectrum $f(a)$. This is a property which could be labelled with the term ergodicity. Note, however, that classical ergodic theory deals with observations along an orbit of increasing length while $f(a)$ is in terms of a sequence of orbits. Also recall that estimating the negative $F(a)$ requires using a large array of realizations of the process (compare remark 3.17).

Partition function:

Since almost all paths of cascades are increasing, the scaling exponents $\alpha_k^{(n)}$ must eventually become positive for any such path and $\tau_\alpha(q)$ must be increasing. The deterministic envelope $T_\alpha(q)$, on the other hand, has a parabolic shape where it is finite. So, $\tau(q) > T(q)$ for $q > (1 + q_{\rm crit})/2$, i.e., for q beyond the maximum of $T(q)$.

Log-normal marginals: The log-normal binomial enjoys the advantage of having *exactly log-normal marginals* $\mu_b(I_k^{(n)})$ since the product of independent log-normal variables is again a log-normal variable.

Finally, we return to the various issues we have left unresolved above.

Negative (virtual) exponents: The most puzzling consequence of the absence of strict conservation of mass resides in the presence of *negative* a with *finite* $T^*(a)$. The apparent paradox of such a was first mentioned and studied in [66]: negative exponents seem to indicate that increments over smaller child-intervals could be larger than the increment over the parent-interval which is impossible for increasing paths.

The paradox was explained away in [66] with the observation that virtual exponents should be interpreted not as a property of the limiting cascade and the actual process, but rather as a property of the iterative cascade construction as the construction progresses, what we *expect* an increment to be over an interval might increase (recall that (5.11) is not the actual increment but its expectation).

Here, we further demystify virtual exponents in providing an interpretation in terms of the actual process, not only its construction. As we have seen above and as follows from corollary 4.7 with more generality we have $F(a) = T^*(a)$ for $a = T'(q)$. For such a $F(a)$ is thus *finite*, and $\mathbb{E}[N^{(n)}(a, \varepsilon)] > 0$. If a is in addition virtual, then $F(a) < 0$ and the expected number of $\alpha_k^{(n)} \simeq a$ decays rather rapidly as resolution n increases but never reaches zero. Thus, even for a large n an increment larger than 1 over an interval of length 2^{-n} can be observed, though extremely rarely, leading to a negative $\alpha_k^{(n)}$. This does not contradict the fact that for any *fixed* path all $\alpha_k^{(n)}$ will eventually (for large n) be positive.

♠

Remark 5.9. (Deterministic envelope: shape and interpretation)

There are several distinct parts of the 'Legendre spectrum' $T^*(a)$.

Looking first at T^*, the positive part (where $T^*(a) > 0$) relates most directly to local properties of individual paths. For cascades, it provides $\dim(E^{[a]})$, the dimension of sets with equal local behavior as measured through $s_k^{(n)}$, as well as $f(a)$ and $\tau^*(a)$; in general it provides at least an upper bound. This part of $T^*(a)$ lies, obviously, between its two zeros which in turn correspond to the slopes of the two tangents at $T(q)$ through the origin (recall that these functions are concave and apply the Legendre formalism (4.7)).

The remaining $T^*(a)$, i.e., the negative ones, can not be interpreted as actual dimensions. However, through the LDP of (4.11) and (3.23) they still provide information on particularly rare degrees a of local scaling. Sometimes they are referred to as "negative dimensions".

Second, let us discuss now the values of a which may appear in the spectrum. The not immediately intuitive ones are the "latent" exponents (a with $T^*(a) < 0$) and the "virtual" ones ($a < 0$). For the scaling exponents $\alpha_k^{(n)}$ and $h_k^{(n)}$ of *increasing* processes all virtual exponents are necessarily latent. This follows from $T(1) = 0$ which implies that $T^*(a) \leq a$ (compare remark 4.6); indeed, no increasing path can show a negative increment-exponent in the limit and virtual exponents can exist here only as 'rare events' (compare remark 3.17). The name 'virtual' has to be understood in this context. For wavelet-based exponents $w_k^{(n)}$ and/or non-increasing processes, however, negative a may occur pathwise and very naturally (compare Section 6.1).

♣

5.6 Multiplicative Schemes beyond Dyadic Structure

The dyadic structure inherent to the binomial cascades can at times be too restrictive. The broader class of 'self-similar cascades' offers more flexibility going beyond dyadic and even towards random subdivision of intervals. Despite this added freedom one can think of cascades in principle as various types of 'muscles' hanging off an always present *tree*-skeleton. Indeed, the tree at the core of all cascades provides common structure and a central tool in the multifractal analysis, enabling an approach similar to the binomial case.

It should not go without mentioning, though, that various technical difficulties arise when dealing with this more general framework, such as with the moments of negative order for cascades with fractal support. With regard to this particular problem we describe here a method which relies on modifying the scaling exponents $h_k^{(n)}$ as in [91]. For wavelet exponents, the way to go is to modify the partition function by employing the lines of maxima, pioneered in [78]. It is notable that both approaches can be considered as a multifractal analysis with new scaling exponents, related to $h_k^{(n)}$ and $w_k^{(n)}$ respectively; consequently, the multifractal formalism of section 4 still holds for these modified methods.

Example 5.4. (Multi-nomial cascades) The first most obvious generalization of the binomial measure is to split intervals into c subintervals. Making the obvious adjustments such as organizing the multipliers now in a c-ary tree and adapting (i)-(iii) (see (5.5)) this leads to a measure μ_c.

Towards understanding the multifractal properties of μ_c choose m such that $2^n \simeq c^m$. Then, a typical interval of length 2^{-n} will have a μ_c-mass approximately equal to $M_0^{(0)} \cdot M_{k_1}^{(1)} \cdots M_{k_m}^{(m)}$ where k_i indicates now position on a c-ary tree. As in (5.25) we find $\mathbb{E}_\Omega[S^{(n)}_{\alpha,\mathcal{M}_c}(q)] \simeq (\mathbb{E}[(M_0)^q + \ldots + (M_{c-1})^q])^m$. Since $m/n \simeq \log(2)/\log(c)$ this leads to

$$T_{\alpha,\mathcal{M}_c}(q) = -\log_c \mathbb{E}[(M_0)^q + \ldots + (M_{c-1})^q] \tag{5.33}$$

This formula is indeed correct for all $q \in \mathbb{R}$ and equality holds in the multifractal formalism provided all the multipliers are zero with probability zero.

Cascades with fractal support and scaling exponents: A true advantage of splitting intervals into more than two subintervals resides in the possibility to leave some of these subintervals without mass: split intervals into $c' > c$ subintervals and distribute mass in the usual multiplicative manner but only among selected c subintervals. This construction can be viewed as an extreme case of a c'-nomial cascade where now certain multipliers are set to zero almost surely. This leads to conjecture that

$$T_{\mathcal{M}_c}(q) = -\log_{c'} \mathbb{E}[(M_0)^q + \ldots + (M_{c-1})^q] \tag{5.34}$$

Due to the construction the limiting measure has a fractal support of dimension $-T(0) = \log(c)/\log(c') < 1$. This might be of practical advantage: when modelling rain data, e.g., the actual events of interest occupy a very small set on the time axis.

It may be 'shocking' news at first sight that equality in the multifractal formalism in the sense of theorem 5.6 holds no longer for cascades with fractal support (see [91]). What breaks down is the partition function: Unless c is a power of 2, (5.34) holds for exponents $h_k^{(n)}$ and $\alpha_k^{(n)}$ only for $q > 0$ and

$$\tau_\alpha(q) = T_\alpha(q) = -\infty \qquad (q < 0) \tag{5.35}$$

due to boundary effects which produce exponents that are exceptionally large, yet finite. As is demonstrated in [91] (see also [84]), a simple remedy for *increasing* processes $\mathcal{M}$ is to exclude the dyadic intervals $I_k^{(n)}$ over which $\mathcal{M}$ is constant and to set

$$\lambda_k^{(n)} := \begin{cases} h_k^{(n)} & \text{if } \alpha_k^{(n)} \neq \infty \\ \infty & \text{otherwise.} \end{cases} \tag{5.36}$$

In other words, $2^{-n\lambda_k^{(n)}}$ is the increment of $\mathcal{M}$ over the interval $I_{k-1}^{(n)} \cup I_k^{(n)} \cup I_{k+1}^{(n)}$ if $\mathcal{M}$ has a strictly positive increase over the middle interval $I_k^{(n)}$; if $\mathcal{M}$ is flat over the middle interval $I_k^{(n)}$ then $2^{-n\lambda_k^{(n)}} = 0$ and there is no contribution of $I_k^{(n)}$ to the partition sum.

Let us check that this procedure is an extension of the usual analysis. First, since limits of $h_k^{(n)}$ and $\lambda_k^{(n)}$ $(n \to \infty)$ are equal we have $E_h^{[a]} = E_\lambda^{[a]}$ etc. Second, if $\mathcal{M}$ is *strictly* increasing then $h_k^{(n)} = \lambda_k^{(n)}$ even for all n and k, and $\tau_\lambda = \tau_h$. Finally, for general cascades and positive q (which correspond to small a of the spectrum) the partition function has not changed either:

Lemma 5.10. **[91]** *Assume that $\mathcal{M}$ is increasing. Then,*

$$\tau_\lambda(q) = \tau_h(q) = \tau_\alpha(q) \qquad (q \geq 0). \tag{5.37}$$

To obtain the multifractal properties of $\mathcal{M}_c$ take expectations in the proofs of [91]. It follows that $T_{\lambda,\mathcal{M}_c}(q)$ is the unique solution of (5.34) for *all* q. From [33, 6, 8] it follows then that the dimension based spectra are equal to T^*_λ where it is positive. In short, equality holds in the multifractal formalism in the following sense: For multi-nomial cascades

$$\dim(K_h^{[a]}) = \dim(K_\lambda^{[a]}) = f_{\lambda,\mathcal{M}_c}(a) = \tau^*_{\lambda,\mathcal{M}_c}(a) \overset{\text{as}}{=} T^*_{\lambda,\mathcal{M}_c}(a) \tag{5.38}$$

wherever the right hand side is positive.

As is shown in [84, 91] using the coarse singularity exponent $\lambda_k^{(n)}$ provides a reliable estimation of the multifractal structure of distributions with fractal support. Using $h_k^{(n)}$ or $\alpha_k^{(n)}$, on the other hand, leads to numerical instability when estimating $\tau(q)$ for negative q. ♠

A more radical way of leaving the binary geometric structure behind is by randomizing it. To this end we consider first the inverse of binomials. Inverse measures can be defined in all generality (see [72, 94]) allowing for further flexibility.

Example 5.5. (Random geometry: Inverse measures) Let $\mathcal{M}_b^\dagger$ be the inverse function of a binomial distribution function $\mathcal{M}_b$, in other words, $\mathcal{M}_b(\mathcal{M}_b^\dagger(t)) = t$.

Then, $\mathcal{M}_b^\dagger$ is almost surely increasing and continuous. So, it defines a random measure $\mu_b^\dagger$ which can be described as follows: for all n, the unit interval is divided into 2^n subintervals $J_k^{(n)}$ of random lengths $M_0^{(0)} \cdot M_{k_1}^{(1)} \cdots M_{k_n}^{(n)}$ and with $\mu_b^\dagger$ mass $1/2^n$ each.

To motivate a formula for $T(q)$ in this case let us note that $2^{n\gamma} \cdot \mathbb{E}[S^{(n)}(q)]$ should tend to 0 for $\gamma > T(q)$, resp. to ∞ for $\gamma < T(q)$. Noting that $2^{n\gamma} = |I_k^{(n)}|^{-\gamma}$, we may equivalently study $\mathbb{E}[\sum \mu_b^\dagger(J_k^{(m)})^q |J_k^{(m)}|^{-\gamma}]$ for finer and finer partitions of $[0,1]$ into intervals $J_k^{(m)}$. Therefore, we iteratively split the largest interval of such a partition, say $J_v^{(u)}$, into its subintervals $J_{2v}^{(u+1)}$ and $J_{2v+1}^{(u+1)}$ which have lengths $Mscaleuo_{2v} \cdot |J_v^{(u)}|$ and $M_{2v+1}^{(u+1)} \cdot |J_v^{(u)}|$. The contribution of $J_v^{(u)}$ to the partition sum was $\mu_b^\dagger(J_v^{(u)})^q |J_v^{(u)}|^{-\gamma}$; the iteration step will replace it by $\mu_b^\dagger(J_{2v}^{(u+1)})^q |J_{2v}^{(u+1)}|^{-\gamma} + \mu_b^\dagger(J_{2v+1}^{(u+1)})^q |J_{2v+1}^{(u+1)}|^{-\gamma}$ which can be written as

$$\mu_b^\dagger(J_v^{(u)})^q |J_v^{(u)}|^{-\gamma} \left(2^{-q}(M_{2v}^{(u+1)})^{-\gamma} + 2^{-q}(M_{2v+1}^{(u+1)})^{-\gamma}\right). \tag{5.39}$$

This leads us to pose that $T_{\mathcal{M}_b^\dagger}(q)$ is the unique solution of

$$\mathbb{E}\left[(1/2)^q (M_0)^{-T(q)} + (1/2)^q (M_1)^{-T(q)}\right] = 1. \tag{5.40}$$

Indeed, during iteration $\mathbb{E}[\sum \mu_b^\dagger(J_k^{(m)})^q |J_k^{(m)}|^{-\gamma}]$ will decay, resp. increase (on the average) according to whether $\gamma > T(q)$ or $\gamma < T(q)$, as required. ♠

Example 5.6. (Statistically self-similar measures) Combining above examples leads to defining a random binomial $\mathcal{M}_c$ by splitting intervals $J_k^{(n)}$ iteratively into c subintervals $J_{ck}^{(n+1)}, \ldots, J_{ck+c-1}^{(n+1)}$ with length $|J_{ck+i}^{(n+1)}| = L_{ck+i}^{(n+1)} |J_k^{(n)}|$ and mass $\mu_c(J_{ck+i}^{(n+1)}) = M_{ck+i}^{(n+1)} \mu_c(J_{ck}^{(n)})$. In the most simple case, one will require conservation of mass, i.e.,

$M_{ck}^{(n+1)} + \ldots + M_{ck+c-1}^{(n+1)} = 1$, but also $L_{ck}^{(n+1)} + \ldots + L_{ck+c-1}^{(n+1)} = 1$ which guarantees that μ_c lives everywhere. Assuming the conditions (ii) and (iii) (see (5.7)) hold for both, the length-multipliers $L_k^{(n)}$ as well as the mass-multipliers $M_k^{(n)}$ we find that $T_{\mathcal{M}_c}(q)$ is the unique solution of

$$\mathbb{E}\big[(M_0)^q (L_0)^{-T(q)} + \ldots + (M_{c-1})^q (L_{c-1})^{-T(q)}\big] = 1. \tag{5.41}$$

A rigorous proof of (5.41) is obtained by 'taking expectations' in the proof of [91, Prop 14]. Doing so shows, moreover, that $T(q)$ assumes a limit in these examples. ♠

In the broader context of Example 5.6 it is easy to recognize that all the examples above are merely special cases of (5.41). This confirms, in particular, (5.40).

Example 5.7. (Cascades in the wavelet domain) As the concluding example we mention that, with regard to (5.14), one may choose to model directly the wavelet coefficients of a process in a multiplicative fashion in order to obtain a desired multifractal structure. First steps in this direction have been taken in [1]. ♠

A final remark is in order.

Remark 5.11. (Stationary increments:) Due to property (iii) the binomial cascade has increments $\mathcal{M}\left((k+1)2^{-n}\right) - \mathcal{M}(k2^{-n})$ which have identical distributions provided M_0 and M_1 are equally distributed. This first order stationarity of dyadic increments is sometimes sufficient to obtain interesting results (see lemma 8.8). On the other hand, these increments are clearly not second order stationary.

However, to obtain true stationarity of increments one has to leave rigid multinomial subdivisions towards stationary ones, e.g., produced by Poisson arrivals. For some appealing examples see [14, 70, 74, 105]. With these models one trades, potentially, appealing statistical properties against more involved analysis. ♣

6 Wavelet based Analysis

In this section we start with a note on the appealing connection between wavelet-based multifractal analysis and classification of functions using regularity spaces such as Besov spaces. Then, we revisit the binomial cascades 3with wavelets and demonstrate a simple relation between the multifractal properties of a process and its (distributional) derivative.

6.1 Wavelet Coefficients and Besov Spaces

Besov spaces are function spaces tailored for studying global regularity. They have become of fashion recently since an elegant description in terms of wavelet coefficients (see [76]) has become available. In [76] it is shown that the norm of the Besov space $B_v^s(L^u)$ of a process with wavelet coefficients $C_{j,k}$ is equivalent, in the notation of (2.7), to

$$\left(\sum_k |D_{0,0}|^v\right)^{1/v} + \left(\sum_{j\geq J_0}\left(\sum_k 2^{jsu}2^{-j}\left|2^{j/2}C_{j,k}\right|^u\right)^{v/u}\right)^{1/v}. \tag{6.1}$$

Roughly speaking, this norm measures the smoothness of order s in L^u, where v is an additional parameter for making finer distinctions in smoothness.

A connection between multifractal theory and Besov spaces has been established in [30]. Here, we note that multifractal analysis (using convenient orthogonal wavelets) can be viewed as determining in which Besov spaces the analyzed process lies [89]. Indeed, by the very definition (3.10) of the partition function with $s_k^{(n)} = w_k^{(n)}$ (see (2.11)) the $B_v^s(L^u)$ norm of a path of the process is finite if $su < \tau_w(u) + 1$ and infinite if $su > \tau_w(u) + 1$ (see [53] for similar results). In other words, pathwise

$$\sup\{s : Y \in B_v^s(L^u)\} = \frac{\tau_w(u) + 1}{u}. \tag{6.2}$$

Strictly speaking s must be smaller than the regularity r of the wavelet ψ for (6.1) and (6.2) to hold. In other words, at least r vanishing moments as well as r continuous derivatives of ψ are required. Given this, Besov norms do not depend on the choice of the wavelet basis up to bounded factors. Since the multifractal analysis using wavelets determines the Besov spaces that contain the signal, we conclude that $\tau_w(u)$ will *not depend on the choice of the wavelet*, provided the above regularity conditions are met.

6.2 The Binomial Revisited with Wavelets

The binomial measure μ_b is not a function or process in the usual sense, but a random distribution. Thus, increment exponents such as $h_k^{(n)}$ and $\alpha_k^{(n)}$ are not well defined for μ_b and we have to rely on wavelet exponents $w_k^{(n)}$ to reveal the local scaling structure. As is often the case, the deterministic envelope is the most simple spectrum to compute. Taking into account the normalization factors in (2.11) when using lemma 5.4, the calculation of (5.24) carries over to give

$$\mathbb{E}[S_{w,\mu_b}^{(n)}(q)] = 2^{nq}\mathbb{E}[|C_{0,0}|^q] \cdot (\mathbb{E}_\Omega[M_0^q] + \mathbb{E}_\Omega[M_1^q])^n .$$

Similar manipulations work for $S_{w,\mathcal{M}_b}^{(n)}$. In summary:

Lemma 6.1. *Provided the wavelet coefficients* $\mathbb{E}[|C_{0,0}(\mu_b)|^q]$ *resp.* $\mathbb{E}[|C_{0,0}(\mathcal{M}_b)|^q]$ *and the moments* $\mathbb{E}[M_0^q]$, $\mathbb{E}[M_1^q]$ *and* $\mathbb{E}[(\mathcal{M}_b(1))^q]$ *are all* finite *we have*

$$T_{w,\mu_b}(q) + q = T_{w,\mathcal{M}_b}(q) = T_{\alpha,\mathcal{M}_b}(q), \tag{6.3}$$
$$T_{w,\mu_b}^*(a-1) = T_{w,\mathcal{M}_b}^*(a) = T_{\alpha,\mathcal{M}_b}^*(a). \tag{6.4}$$

Imposing additional assumptions on the distributions of the multipliers we may also control $w_k^{(n)}(\mu_b)$ themselves and not only their moments. To this end, we should be able to guarantee that the (Haar) wavelet coefficients don't decay too fast (compare (2.10)), i.e. the random prefactor RHS in (5.13) and (5.14) doesn't become too small. Indeed, it is sufficient to assume that there is some $\varepsilon > 0$ such that $|C_{0,0}| \geq \varepsilon$ almost surely. (recall that $C_{0,0} = 2M_0 - 1$ for the Haar wavelet.) Then for all t, $(1/n)\log(\int \psi\mu_b^{(n,k_n)}(\mathrm{d}t)) \to 0$, and with (5.14)

$$w_{\mu_b}(t) = \liminf_{n\to\infty} -\frac{1}{n}\log_2\left(2^{n/2}|C_{n,k_n}|\right) = \alpha_{\mathcal{M}_b}(t) - 1, . \tag{6.5}$$

This relation agrees with (6.4) and appeals to one's intuition: the singularity exponents of a 'sufficiently nice' process and its (distributional) derivative should differ by 1, as is demonstrated here with binomial cascades. We should point out, though, that this simple rule-of-thumb may fail miserably in the presence of 'chirps', even for increasing processes. More precise information on how scaling properties change locally —such as provided by the 'two-microlocalization' analysis [51, 50]— is required to fully understand how taking derivatives affects the local Hölder regularity (see also Section 6.3).

Corollary 6.2. *Assume that μ_b is a binomial measure satisfying (i)-(iii) (see (5.5)). Assume furthermore, that the random variables $A_{n,k}(\mu_b)$ resp. $A_{n,k}(\mathcal{M}_b)$ are uniformly bounded away from 0. Then, equality holds in the multifractal formalism for the wavelet based spectra of μ_b, resp. $\mathcal{M}_b$, meaning that*

$$\dim(K^{[a]}_{w,\mu_b}) \overset{\text{a.s.}}{=} f_{w,\mu_b}(a) \overset{\text{a.s.}}{=} \tau^*_{w,\mu_b}(a) \overset{\text{a.s.}}{=} T^*_{w,\mu_b}(a) = T^*_{\alpha,\mathcal{M}_b}(a+1), \tag{6.6}$$

respectively

$$\dim(K^{[a]}_{w,\mathcal{M}_b}) \overset{\text{a.s.}}{=} f_{w,\mathcal{M}_b}(a) \overset{\text{a.s.}}{=} \tau^*_{w,\mathcal{M}_b}(a) \overset{\text{a.s.}}{=} T^*_{w,\mathcal{M}_b}(a). \tag{6.7}$$

For deterministic binomials τ_{w,μ_b} has been calculated in [7]. Note that for the Haar wavelet $A_{n,k} = M^{(n+1)}_{2k} - M^{(n+1)}_{2k+1} = 2M^{(n+1)}_{2k} - 1$.

Requiring that $A_{n,k}$ should be bounded away from zero in order to ensure (6.5), though satisfied in some simple cases, seems unrealistically restrictive to be of practical use. A few remarks are in order, then. First, this condition can be weakened to allow arbitrarily small values of $A_{n,k}$ as long as all its negative moments exist. This can be shown by an argument using the Borel-Cantelli lemma. Furthermore, if one drops (iii) and allows the distributions of the multipliers to depend on scale (compare Example 7.2), then $A_{n,k}$ has to be bounded away from zero only for large n. In applications such as network traffic modelling one finds indeed that on fine scales $A_{n,k}$ is best modelled by *discrete* distributions on $[-1, 1]$ with large variance and no mass around 0.

Another way out is to avoid small wavelet coefficients at all in a multifractal analysis. More precisely, one would follow [7, 53] and replace C_{n,k_n} in the definition of $w^{(n)}_{k_n}$ (2.11) by the maximum over certain wavelet coefficients 'close' to t. Since this procedure amounts to nothing more than introducing a yet better adapted scaling exponent $\tilde{w}$, the multifractal formalism still holds in the sense of Section 4. Towards establishing equality between spectra [53] gives conditions under which the spectrum $\tau^*_{\tilde{w},\mu_b}(a)$ based on this new exponent $\tilde{w}$ agrees with the 'classical Hölder exponent spectrum' $\dim(E^{[a]})$ based on $h^{(n)}_k(\mathcal{M}_b)$.

6.3 Multifractal Properties of the Derivative

Corollary 6.2 establishes for the binomial cascade what intuitively appears to be of general truth: taking derivatives should decrease the local Hölder regularity by 1. This section is devoted to a few words of warning in this context. Let us start with a simple example where the given intuitive rule of thumb fails utterly and continue then to discuss a practical but powerless approach to studying the multifractal properties of the derivative.

As indicated in the last section, the above simplistic rule of thumb fails in the presence of chirps [51]. For an example consider the function $\mathcal{M}(t) = 5t + t^3\sin(1/t)$. Since its derivative $\mathcal{M}'$ is strictly positive, this is a strictly increasing function. Furthermore, the local Hölder regularity of $\mathcal{M}$ at $t=0$ is of order 3, i.e., $H_{\mathcal{M}}(0)=3$, since $|\mathcal{M}(t)-5t| \leq |t|^3$ is the best polynomial approximation in the sense of (2.1). Its derivative, however, shows $H_{\mathcal{M}'}(0)=1$ since it can be approximated as $|\mathcal{M}'(t)-5| \leq |t|$. The local regularity dropped, thus, by 2. This is due to the strong oscillations, also called a *chirp*, present at $t=0$. To fully understand and assess how the differential operator affects the local Hölder regularity, one has to get acquainted with the 'two-microlocalization' analysis [51, 50].

Now, to explore the role of wavelets in this matter we place ourselves in the framework where one cares less for a wavelet representation but where one's interests lie solely in an analysis of the oscillatory behavior. Therefore, we will employ *analyzing* mother wavelets ψ which are typically not orthogonal, such as the derivatives of the Gaussian kernel $\exp(-t^2/2)$ which were used to produce Figure 7.

To connect wavelet coefficients of a process and its derivative we use integration by parts. For a continuous measure μ on $[0,1]$ with distribution function $\mathcal{M}(t)=\mu([0,t))$ and a continuously differentiable function g this reads as

$$\begin{aligned}
\int g(t)\mu(\mathrm{d}t) &= \lim_{n\to\infty}\sum_{k=0}^{2^n-1} g(k2^{-n})\big(\mathcal{M}((k+1)2^{-n})-\mathcal{M}(k2^{-n})\big)\\
&= \lim_{n\to\infty}\sum_{k=0}^{2^n-1} \mathcal{M}(k2^{-n})\big(g((k-1)2^{-n})-g(k2^{-n})\big)\\
&\qquad +\mathcal{M}(1)g(1-2^{-n})-\mathcal{M}(0)g(-2^{-n})\\
&= \mathcal{M}(1)g(1)-\mathcal{M}(0)g(0)-\int \mathcal{M}(t)g'(t)\mathrm{d}t
\end{aligned} \tag{6.8}$$

where we alluded to (5.3) and regrouped terms. If $\mathcal{M}$ has a derivative, then $\mu(\mathrm{d}t)$ becomes $\mathcal{M}'(t)\mathrm{d}t$ and (6.8) reduces to the usual rule of integration by parts. Now, setting $g(t)=2^{n/2}\psi(2^nt-k)$ for a smooth analyzing wavelet ψ we have $g'(t)=2^{3n/2}\psi'(2^nt-k)$ and using that $\mathcal{M}(1)=1$ and $\mathcal{M}(0)=0$, we get

$$C_{n,k}(\psi,\mu)=2^{n/2}\psi(2^n-k)-2^n\cdot C_{n,k}(\psi',\mathcal{M}). \tag{6.9}$$

Estimating $2^n-k_n=2^n-\lfloor t2^n\rfloor \simeq (1-t)2^n$ and assuming exponential decay of $\psi(t)$ at infinity allows to conclude

$$w_{\psi,\mu}(t)=-1+w_{\psi',\mathcal{M}}(t). \tag{6.10}$$

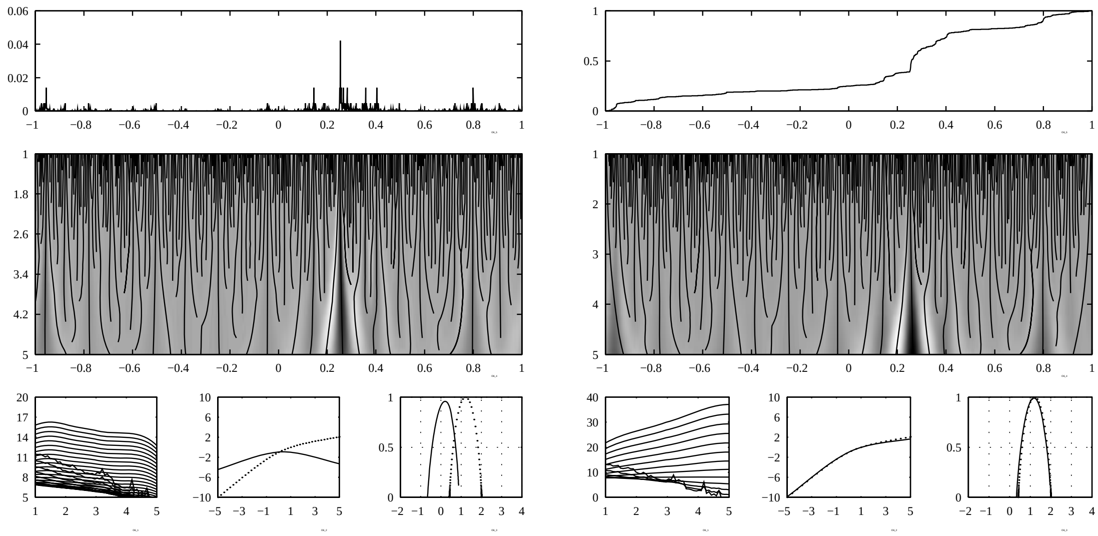

FIGURE 7. *Demonstration of the multifractal behavior of a binomial measure μ_b (left) and its distribution function $\mathcal{M}_b$ (right). On the top a numerical simulation, i.e., (5.2) on the left and $\mathcal{M}_b(k2^{-n})$ on the right for $n = 20$. In the middle the moduli of a continuous wavelet transform [23] where the second derivative of the Gaussian was taken as the analyzing wavelet $\psi(t)$ for μ_b, resp. the third derivative ψ' for $\mathcal{M}_b$. The dark lines indicate the 'lines of maxima' [53, 7], i.e., the locations where the modulus of $\int \psi(2^j t - s)\mu_b(\mathrm{d}t)$ has a local maximum as a function of s with j fixed. On the bottom a multifractal analysis in three steps. First, a plot of $\log S_w^{(n)}(q)$ against n tests for linear behavior for various q. Second, the partition function $\tau(q)$ is computed as the slopes of a least square linear fit of $\log S^{(n)}$. Finally, the Legendre transform $\tau^*(a)$ of $\tau(q)$ is computed following (4.7). Indicated with dashes in the plots of $\tau(q)$ and $\tau^*(a)$ of μ_b are the corresponding function for $\mathcal{M}_b$, providing empirical evidence for (6.3), (6.4), and (6.10).*

Corollary 6.3.

$$E^{[a+1]}_{\psi,\mu} = E^{[a]}_{\psi',\mathcal{M}} \qquad f_{\psi,\mu}(a) = f_{\psi',\mathcal{M}}(a+1), \qquad \tau^*_{\psi,\mu}(a) = \tau^*_{\psi',\mathcal{M}}(a+1). \tag{6.11}$$

This is impressively demonstrated in Figure 7. We should note that ψ' has one more vanishing moment than ψ which is easily seen by integrating by parts. Thus, it is *natural* to analyze the integral of a process, here the distribution function $\mathcal{M}$ of the measure μ, using ψ' since the degree of the Taylor polynomials of $\mathcal{M}$ are typically by 1 larger than the ones of μ, and the analyzing wavelet should be blind to these polynomials.

In the light of the fact that (6.11) holds even in the presence of chirps, the *lesson* to retain is that the multifractal analysis of $\mu = \mathcal{M}'$ through ψ is *equivalent* to the analysis of $\mathcal{M}$ through ψ' and can not serve to discover chirps. It is notable, then, that corollary 6.2 uses the *same* wavelet for both, μ_b and $\mathcal{M}_b$.

Remark 6.4. (Visibility of singularities and regularity of the wavelet)

It is remarkable that the Haar wavelet provides us with the full spectra of both, the binomial $\mathcal{M}_b$ and its distributional derivative μ_b. This seems to contradict rules of thumb saying that a wavelet cannot detect degrees of regularity larger than the number of vanishing moments. In [78] it is even advocated that the partition function $\tau(q)$ should be used only for $q \in [0, 1]$.

To resolve the apparent paradox recall the peculiar property of multiplicative measures which is to have constant Taylor polynomials in the points of interest. So, cascades will reveal their scaling structure to any analyzing wavelet with $\int \psi = 0$. No higher regularity, i.e., $\int t^k \psi(t) \mathrm{d}t = 0$ is required despite the fact that points of regularity $\alpha(t) > 1$ are present. For another example of a signal which is more regular than the basis elements it is composed of see [55].

The mentioned rule has, then, to be corrected to read as: wavelet cannot detect singularities in points with approximating polynomial P_t of degree higher than its regularity. ♣

7 Self-similar Processes

So far we have studied multiplicatively constructed processes which exhibit particular rescaling properties such as the tree structure of the binomial cascade. Let us turn now to the related concept of so-called 'statistical self-similarity'.

7.1 Statistical Self-similarity

For details on the following short introduction to self-similar processes see [100] or [97, p 309]. Here is some notation used throughout: I is either $\mathbb{R}$, $\mathbb{R}_+ = \{t \geq 0\}$ or $\{t > 0\}$; V is a self-similar process with increments U.

Definition 7.1. *A real-valued stochastic process* $\{V_t : t \in I\}$ *is called* self-similar with index $H > 0$ *if and only if for all* $a > 0$

$$V(at) \stackrel{\mathrm{fd}}{=} a^H V(t), \tag{7.1}$$

in the sense of finite dimensional distributions, in other words, if and only if for any $a > 0$*, any* $m \in \mathbb{N}$ *and any* $t_1 < t_2 <, \ldots, < t_m$

$$(V(at_1),, \ldots,, V(at_m)) \stackrel{\mathrm{d}}{=} (a^H V(t_1),, \ldots,, a^H V(t_m)).$$

$\{V(t)\}$ *is said to have* stationary increments *if and only if*

$$V(s+t) - V(t) \stackrel{\mathrm{fd}}{=} V(s) - V(0).$$

A complex valued process is self-similar if and only if both, real and imaginary part are. If a process is both self-similar and has stationary increments we say it is H-sssi.

It follows from the definition that $V(0) = 0$ a.s. for any self-similar process. Self-similar processes themselves are not stationary; as a matter of fact, $V(t)$ is self-similar if and only if $e^{-tH}V(e^t)$ is stationary. It is helpful, therefore, to add the assumption of stationary increments. As an immediate consequence of the definitions, an H-sssi process is centered, and so are its increments, unless it has infinite expectation:

$$\mathbb{E}V(1) = \mathbb{E}[V(1) - V(0)] = \mathbb{E}[V(2) - V(1)] = (2^H - 1)\mathbb{E}V(1).$$

7.2 Examples

If one assumes finite variance, then the covariance is easily computed using (7.1) and stationarity of increments as follows

$$\begin{aligned} r(s,t) = \mathbb{E}V(s)V(t) &= \frac{1}{2}\left(\mathbb{E}V(t)^2 + \mathbb{E}V(s)^2 - \mathbb{E}(V(t) - V(s))^2\right) \\ &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})\mathbb{E}V(1)^2. \end{aligned} \tag{7.2}$$

Since this expression must be non-negative definite it follows that $0 < H \leq 1$. For $H = 1$ one finds

$$\mathbb{E}(V(t) - tV(1))^2 = \mathbb{E}V(t)^2 - 2t \cdot \mathbb{E}V(t)\mathbb{E}V(1) + t^2\mathbb{E}V(1)^2 = 0$$

and, thus, $V(t) = tV(1)$ a.s. for all t. Often one studies the increments of an H-sssi process. Therefore, let us set

$$U_k := V(k+1) - V(k).$$

Here, we choose the fixed lag or step 1 for convenience only. We find, using the expression for $r(s,t)$

$$r_U(k) := \mathbb{E}_\Omega[U(n+k)U(n)] = \frac{\mathbb{E}_\Omega[V(1)^2]}{2}\left(|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H}\right). \tag{7.3}$$

Gaussian case

As mentioned before, the auto-correlation function (7.2) has to be non-negative definite which restricts H to $(0,1]$. On the other hand, for every such function r there is a zero mean Gaussian process defined through its finite dimensional distributions [97, p 318]:

$$\mathbb{E}[\exp i \sum_{k=1}^{m} \theta_k V(t_k)] = \exp\left(-\frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m} r(t_k, t_l)\theta_k\theta_l\right).$$

It can be shown that this process is unique and that it is always possible to choose a version with almost surely continuous paths.

Definition 7.2. *The* unique Gaussian and H-sssi process *with* $0 < H \leq 1$ *is called* fractional Brownian motion *(fBm). We will denote it by* B_H. *Its increment processes* $\{B(t+\delta) - B(t) \,:\, t \in I\}$ *are called* fractional Gaussian noise *(fGn).*

The case $H = 1/2$ is notable in this context as it is the only case where the increments are independent. $\{B_{1/2}(t)\}$ is called Brownian motion, or, to make a clear distinction, *Wiener motion* (WM). FBm has been introduced by Kolmogorov and studied by Lévy, and Mandelbrot and van Ness [71] who gave the following representation as a stochastic integral over Wiener motion

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_{-\infty}^{\infty} [(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}] \mathrm{d}B_{1/2}(s), \tag{7.4}$$

where $(x)_+ = \max(x, 0)$.

Of particular interest here are the fractal properties of the paths of fBm. It is well known that its paths assume almost surely everywhere the exact Hölder regularity of degree H:

Theorem 7.3 (Adler). *Let $[a, b]$ be a compact interval.*
a) [5, Thm. 8.3.1.] For every $\eta > 0$ there is an almost surely finite, positive random variable A such that with probability one for every $s, t \in [a, b]$ and $|s - t|$ small enough

$$|B_H(t) - B_H(s)| \leq A|s - t|^{H-\eta}$$

b) [5, (8.8.26)] For each $\eta > 0$ and small enough $h > 0$

$$\sup\{|B_H(t) - B_H(s)| \,:\, |s - t| \leq h\} \geq K h^{H+\eta}$$

for every $t \in [a, b]$ and every $K < \infty$, with probability one.

Stable case

If we drop the assumption of finite variance we fall on the γ-stable, H-sssi processes as the next larger class. A symmetrical γ-stable random variable V, in short $V \sim S\gamma S(\sigma)$, is defined through its characteristic function:

$$\mathbb{E}[\exp(i\theta V)] = \exp(-\sigma^\gamma |\theta|^\gamma + i\mu\theta) \qquad \text{for } 0 < \gamma \leq 2.$$

It follows, that the sum $V + V'$ of independent $S\gamma S$ variables is again $S\gamma S$ with $\sigma_{V+V'}^\gamma = \sigma_V^\gamma + \sigma_{V'}^\gamma$.

In general, no closed form for the distribution function is known. Special known cases include the Normal distribution ($\gamma = 2$) and the Cauchy distribution ($\gamma = 1$). It is known, though, that the tails decay with a power-law for $0 < \gamma < 2$, i.e. there is a constant C_γ depending only on γ such that [97, p. 16]

$$\lim_{\lambda\to\infty} \lambda^\gamma P[V > \lambda] = \lim_{\lambda\to\infty} \lambda^\gamma P[V < -\lambda] = C_\gamma \cdot \sigma^\gamma. \tag{7.5}$$

Consequently, the absolute moments of V exist only up to order γ. For $q > 0$: $\mathbb{E}|V|^q < \infty$ if and only if $q < \gamma$. For $\gamma = 2$, the distribution is a Gaussian which has finite moments of all positive orders.

Let us denote H-sssi, $S\gamma S$-stable processes by $L_{H,\gamma}$. Applying the definitions we find

$$L_{H,\gamma}(t) - L_{H,\gamma}(s) \sim S\gamma S(|t-s|^H \cdot \sigma_{V(1)})$$

Since we may write $L(t) - L(s) + L(s) - L(u) = L(t) - L(u)$, the increments will be independent if and only if $H = 1/\gamma$, which generalizes the Gaussian case $\gamma = 2$. A H-sssi, $S\gamma S$-stable process with $H = 1/\gamma$ is called *Lévy stable motion* and we denote it by $L_H := L_{H,1/H}$.

Finally, it can be shown that necessarily $H \le \max(1/\gamma, 1)$. Except for the Gaussian case $\gamma = 2$ the H-sssi, γ-stable motion are not unique and their paths are discontinuous with probability one. For $H < 1$, a class of fractional stable motions, so-called *linear fractional stable motions*, can be constructed explicitly through integral representations similar to (7.4), where the exponents of the integration kernel are now $H - 1/\gamma$ instead of $H - 1/2$ and where the integrating process is Lévy stable motion L_H instead of Wiener Brownian motion $B_{1/2}$.

7.3 Multifractal Properties of Self-similar Processes

Deterministic Envelopes

For an H-sssi process $V(t)$ we find, due to $V(0) = 0$,

$$2^{-n\alpha_k^{(n)}} = |V((k+1)2^{-n}) - V(k2^{-n})| \overset{\mathrm{d}}{=} |V(2^{-n})| \overset{\mathrm{d}}{=} 2^{-nH}|V(1)| = 2^{-nH}2^{-\alpha_0^{(0)}}. \quad (7.6)$$

More generally:

Lemma 7.4. *For an H-sssi process, the singularity exponents $\alpha_k^{(n)}$, $h_k^{(n)}$, and $w_k^{(n)}$ obey the scaling law*

$$s_k^{(n)} \overset{\mathrm{d}}{=} H + (1/n)s_0^{(0)} \qquad (s \in \{\alpha, h, w\}). \quad (7.7)$$

Furthermore, $\mathbb{E}_\Omega[2^{-qs_0^{(0)}}]$ is finite exactly for $\underline{q} < q < \overline{q}$, where for $\alpha_k^{(n)}$ and $w_k^{(n)}$

$$(\underline{q}, \overline{q}) = \begin{cases} (-1, \infty) & \text{for fBm,} \\ (-1, \gamma) & \text{for } S\gamma S \text{ motion,} \end{cases} \quad (7.8)$$

and for $h_k^{(n)}$

$$\begin{aligned} \overline{q} &= \infty && \text{for fBm,} \\ \overline{q} &\le \gamma && \text{for } S\gamma S \text{ motion,} \\ \underline{q} &= -\infty && \text{for Lévy stable and Wiener motion.} \end{aligned} \quad (7.9)$$

Proof

Property (7.7) follows for $\alpha_k^{(n)}$ from (7.6), and similarly for $h_k^{(n)}$ using continuity of paths. For $w_k^{(n)}$ see [24]. Property (7.8) is well-known (compare (7.5)) since $2^{-n\alpha_k^{(n)}}$ and $2^{-nw_k^{(n)}}$ are the moduli of Gaussian, resp. stable variables [24].

Next, let us discuss $\overline{q}$ for $h_k^{(n)}$. For SγS-stable motion the obvious estimate $2^{-n\alpha_k^{(n)}} \le 2^{-nh_k^{(n)}}$ implies immediately that $\overline{q} \le \gamma$. For fBm we note that [59, lem. 12.2.1] actually shows that the tail probability $P[\sup_{t\in[0,1]}|B_H(t)|>x]$ is smaller than $4\exp(-cx^2)$, which implies $\overline{q}=\infty$.

Finally, still for $h_k^{(n)}$, $\underline{q}=-\infty$ can be derived at least for Lévy stable motion L_H and Wiener motion $B_{1/2}=L_2$ using stationarity and independence of increments as follows:

$$\begin{aligned} P[\sup_{0\le t\le 1}|L_H(t)|\le\varepsilon] &\le P[|L_H(\tfrac{k+1}{m})-L_H(\tfrac{k}{m})|\le 2\varepsilon \quad \text{for } k=0,\ldots,m-1] \\ &= P[|L_H(1/m)|\le 2\varepsilon]^m \le (cm^H\cdot\varepsilon)^m \end{aligned}$$

for some constant c and all m (for the Gaussian case see also [48]). This is sufficient to show that all moments of negative order exist. We believe that $\underline{q}=-\infty$ also for fBm. ♢

Using (7.7) one finds immediately

$$\mathbb{E}_\Omega\left[S^{(n)}(q,\omega)\right]=\mathbb{E}_\Omega\sum_{k=0}^{2^n-1}2^{-nqs_k^{(n)}}=2^n2^{-nHq}\mathbb{E}_\Omega\left[2^{-qs_0^{(0)}}\right],$$

and, with (7.8) and (7.9)

$$T_s(q):=\lim_{n\to\infty}\frac{\log_2\mathbb{E}_\Omega S^{(n)}(q)}{-n}=\begin{cases} qH-1 & \text{for } \underline{q}<q<\overline{q} \\ -\infty & \text{else.}\end{cases} \tag{7.10}$$

For the computation of $F(a)$ let us denote the density function¶ of the distribution of $2^{-s_0^{(0)}}$ by Φ_s. The H-sssi property (see lemma 7.4) and the mean value theorem yield

$$\begin{aligned} P_\Omega[|s_k^{(n)}-a|<\varepsilon] &= P_\Omega[n(a-H-\varepsilon)<s_0^{(0)}<n(a-H+\varepsilon)] \\ &= \int_{2^{-n(a-H+\varepsilon)}}^{2^{-n(a-H-\varepsilon)}}\Phi_s(x)\mathrm{d}x \\ &= 2^{-n(a-H-\varepsilon)}(1-2^{-2n\varepsilon})\Phi_s\left(2^{-n(x_{a,n}-H)}\right) \end{aligned} \tag{7.11}$$

where $x_{a,n}\in[a-\varepsilon,a+\varepsilon]$. It is instructive to compare this to the rescaling property of the marginals of the log-normal binomial cascade (5.32).

Applying lemma 7.4 $\Phi_s(x)$ behaves as $x^{-\underline{q}-1}$ at zero, resp. as $x^{-\overline{q}-1}$ at ∞. From this the asymptotic behavior of (7.11) follows immediately provided we know whether $x_{a,n}-H$ is eventually positive or negative. This observation resolves the cases $a-\varepsilon>H$ and $a+\varepsilon<H$. To cover the case $a=H$ let us note that trivially $P[2^{-n\varepsilon}<|V(1)|<2^{n\varepsilon}]\to 1$. In summary,

$$\frac{1}{n}\log_2 P_\Omega[|s_k^{(n)}-a|<\varepsilon]=\begin{cases}(a-H)\underline{q}+o(\varepsilon,n) & \text{if } a>H+\varepsilon \\ o(\varepsilon,n) & \text{if } a=H \\ (a-H)\overline{q}+o(\varepsilon,n) & \text{if } a<H-\varepsilon.\end{cases} \tag{7.12}$$

¶Recall that $2^{-\alpha_0^{(0)}}=|V(1)|$, $2^{-h_0^{(0)}}=\sup_{[-1,2]}|V(t)|$ etc.

where $o(\varepsilon, n) \le (1/n)\log_2(1-2^{-2n\varepsilon}) + c\cdot\varepsilon$ for sufficiently large n. In the case $a < H-\varepsilon$, e.g., we may choose $c = 2\overline{q}+4$. From this, a formula for $F(a)$ follows immediately. Moreover, (7.12) implies also $F(a) = \underline{F}(a)$ since $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ may differ at the most by $c\varepsilon$. Infinite values for the partition function and the straight parts of the resulting Legendre transform where mentioned in [66, 77].

Corollary 7.5. *For an H-sssi process the deterministic partition function obtained from the coarse singularity exponents $\alpha_k^{(n)}$, $h_k^{(n)}$ or $w_k^{(n)}$ is given by (see Figure 8)*

$$T_s^*(a) = \begin{cases} 1+\overline{q}(a-H) & \text{for } a < H \\ 1 & \text{for } a = H, \\ 1+\underline{q}(a-H) & \text{for } a > H. \end{cases} \tag{7.13}$$

Thereby, $\overline{q}$ and $\underline{q}$ are provided in lemma 7.4. Moreover, $F(a) = \underline{F}(a)$ (compare (3.24)) and

$$F_\alpha = F_w = T_\alpha^* = T_w^*, \qquad F_h = T_h^*. \tag{7.14}$$

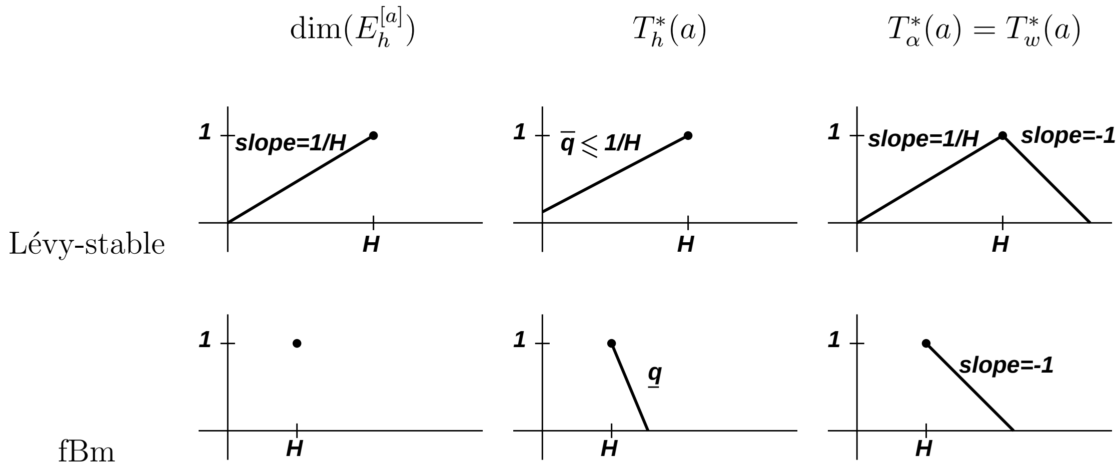

FIGURE 8. *The theoretical spectra of some self-similar processes. Top row: $1/H$-stable Lévy motion (H-sssi) bottom row: fractional Brownian motion B_H (H-sssi)*

Dimension based Spectra

For fBm a result by Adler [5] (compare theorem 7.3) states that with probability one $h(t) = H$ for all t. So, for $a \le H$ we have almost sure equality of the spectra:

$$\text{fBm:} \qquad \dim(E_h^{[a]}) = f_h(a) = \tau_h^*(a) = T_h^*(a) = \begin{cases} 1 & a = H, \\ -\infty & a < H. \end{cases} \tag{7.15}$$

In this light, it is appropriate to call fBm *mono-fractal*. For the interested reader we mention that it is possible to find multifractal structure in the occupation measure of Brownian and stable motion, however not in terms of pointwise exponents but pointwise densities (see [25] and references therein).

For Lévy stable motion L_H (H-sssi, $\gamma = 1/H$-stable process, $\gamma < 2$) Jaffard citeJ5 computed the dimension spectrum (compare Figure 8 and (7.17)):

$$\text{Lévy motion:} \qquad \dim(E_h^{[a]}) = \begin{cases} a/H & 0 < a < H, \\ -\infty & \text{else.} \end{cases} \tag{7.16}$$

Grain based Spectra

Wiener Brownian motion $B_{1/2}$ as well as Lévy stable motion ($H = 1/\gamma$) have independent increments. Thus, corollary 7.5, theorem 3.14 and corollary 4.1 combine to yield almost surely

$$\text{Lévy motion:} \qquad \underline{f}_s(a) = f_s(a) = F_s(a) = \tau_s^*(a) = T_s^*(a) \tag{7.17}$$

for all a for which $T^*(a) \geq 0$ or $= -\infty$. This is valid for the singularity exponents $\alpha_k^{(n)}$, $h_k^{(n)}$ as well as for $w_k^{(n)}$ for compactly supported wavelets; (7.13) provides a formula of T^*. Counting on the strong decorrelation of wavelet coefficients of fBm [35, 57] and keeping in mind remark 3.15 one may hope to find the same result true for fBm with $H \neq 1/2$. Numerical evidence supporting this point is found in [41].

Partition Function

Due to the linearity of $T(q)$ we are able to compute $\tau(q)$ for any H-sssi process assuming only that $T(q)$ is *finite for some negative and some positive* q. We note first that for each n, $S^{(n)}(0, \omega)$ counts the number of non-vanishing increments which is 2^n almost surely. Thus, $\tau(0) = -1$ almost surely. Similarly, $T(0) = -1$ always. Now observe that $\tau(q)$ must be concave, yet lie 'above' $T(q)$ due to lemma 3.9. This is only possible if $\tau(q)$ is linear as well. In conclusion, if $\underline{q} < 0 < \overline{q}$, then with probability one:

$$H\text{-sssi:} \qquad \tau(q, \omega) = qH - 1 \qquad \text{for all } \underline{q} < q < \overline{q}. \tag{7.18}$$

Almost sure bounds can be derived for other q values using that $\tau(q)$ is concave and that $\tau^*(a)$ is positive (compare [62]).

Failure of the multifractal formalism?

A stark discrepancy in the decreasing part between the local spectrum based on Hölder exponents $\dim(E_h^{[a]})$ (often considered the 'true spectrum') and the deterministic envelope $T_{\alpha,w}^*$ based on increment or wavelet exponents $\alpha_k^{(n)}$ or $w_k^{(n)}$, respectively, is immediately apparent from Figure 8. One could call this a failure of $T_{\alpha,w}^*$ to spot the true Hölder continuity. Technically speaking, the reason for this feature is the fact $\alpha_k^{(n)}$ and $w_k^{(n)}$ are centered (zero-mean) random variables of ω, actually Gaussian, resp. stable variables with considerable probability around zero.

Keeping in mind (7.17) and (7.18), however, one has to acknowledge that $T_{\alpha,w}^*$ nevertheless contain crucial scaling information which is relevant in real world application. The part of these spectra forming the line of slope -1 represent the considerably high chance to observe large $\alpha_k^{(n)}$ and $w_k^{(n)}$ at resolution 2^{-n}, i.e., increments and wavelet coefficients of magnitude smaller than 2^{-nH}.

In order to obtain information on the true Hölder continuity from wavelet exponents $w_k^{(n)}$ one has to employ the lines of maxima (see [78, 79, 7, 23]). Taking local maxima of the modulus of wavelet coefficients one avoids exceptionally small $w_k^{(n)}$; the corresponding partition function comes then closer to τ_h since less mass in concentrated around zero, allowing for more moments of negative order to be finite. The interested reader might want to compare this remedy with the use of $\lambda_k^{(n)}$ which avoids exceptionally small increments for measures on fractal support (5.36).

In conclusion, one should keep in mind this *artifact* of multifractal analysis based on data such as increments which is of *zero mean*. Though some information on the local singularity structure of the process might be hidden to such an analysis, the resulting T^* nevertheless corresponds to true scaling behavior of the process in the sense of (7.17) which might be relevant. In this context, it becomes essential to detect divergence of moments from finite data, an issue which is under investigation at present [39].

7.4 Long-Range Dependence (LRD)

The rigid correlation structure of fGn (7.3) is somewhat restrictive for modelling purposes. However, the fact that its auto-correlation decays very slowly has been found to be an important feature in itself and has inspired weaker notions of "similarity on all scales" in terms of second-order statistics only. The novice reader should avoid a common confusion with these weaker notions and be aware that they are defined in terms of the increment processes on varying lags rather than in terms of the process Y itself.

Second Order Scaling

It is easy to see that (7.3) decays like $r_U(k) \simeq k^{2H-2}$. For $1/2 < H < 1$, the correlation is strictly positive and decays so slowly that it is not sumable over k. A process X – which we think of as being the increment process of Y – with this property ($\sum_k r_X(k) = \infty$) is said to exhibit *long range dependence* (LRD), since it exhibits strong correlations at large lags. LRD can be equivalently characterized in terms of the behavior of the aggregated processes

$$X^{(m)}(k) := \frac{1}{m} \sum_{i=km}^{(k+1)m-1} X(i). \tag{7.19}$$

As [19] shows, $r_X(k) \simeq k^{2H-2}$ is equivalent to $\mathrm{var}(X(k)) \approx m^{2-2H}\mathrm{var}(X^{(m)}[n])$. So, we are lead to define:

Definition 7.6. *We say that X is* asymptotically second order self-similar *with parameter H_{var} if the following limit exists:*

$$H_{\mathrm{var}} := 1 + \frac{1}{2} \lim_{m\to\infty} \frac{\log\left[\mathrm{var}(X^{(m)})/\mathrm{var}(X^{(1)})\right]}{\log m} \tag{7.20}$$

As an example let us mention the increments of an H-sssi process V (of finite variance): setting $X(k) = U(k) = V(k+1) - V(k)$ one finds $H_{\mathrm{var},X} = H$ due to (7.1) or (7.6) (compare (1.17)).

Alternatively, LRD can be measured through spectral properties, since $r_X(k) \simeq k^{2H-2}$ is equivalent to a power-law behavior of its Fourier transform, the power spectrum, i.e., $\Gamma_X(\nu) = \mathcal{F}(r_X)(\nu) \simeq \nu^{-(2H-1)}$. The spectral estimation using the wavelet transform of X developed in [4] translates this into

$$\mathrm{var}(C^X_{j,k}) \propto 2^{-j(2H-1)}. \tag{7.21}$$

One may adopt the following terminology:

Definition 7.7. *X is* asymptotically wavelet self-similar *with parameter H_{wave} if the following limit exists:*

$$H_{\mathrm{wave}} := \frac{1}{2} - \frac{1}{2}\lim_{j\to\infty}\frac{1}{j}\log_2 \mathrm{var}(C_{j,k}). \tag{7.22}$$

Also, H_{wave} can be considered an estimator of H for H-sssi processes. Similar as before, $H_{\mathrm{wave}} = H$ for fGn [35, 57]. It should be noted here that the wavelet coefficients of fGn are typically much less correlated than those of the underlying sampled fBm process. Thus, wavelet-based estimators for H compare favorably with standard estimation techniques [4] and is particularly superior to H_{var} [99].

Multifractal Envelope and LRD

The multifractal scaling exponent $T(2)$ of a process Y is closely related to the LRD parameter H, since both measure the power-law behavior of second-order statistics. More precisely, $T(2)$ captures the scaling behavior of the second sample moments, while H captures the decay of the covariances.

From Section 1.5 we recall:

Lemma 7.8. *For a process Y with zero-mean increments*

$$H_{\mathrm{var}} = \frac{T_{\alpha,Y}(2) + 1}{2}. \tag{7.23}$$

For fBm, this is in agreement with (7.13).

Multifractal processes defined through multiplication on a tree such as the binomial $X(k) = \mu_b(I^{(n)}_k)$ are not second-order stationary. Hence, LRD cannot be defined through the decay of the auto-covariances. However, alternative fractal properties, such as the decay of aggregate variances (7.20) or wavelet coefficients (7.22) — which are equivalent to LRD in the presence of second-order stationarity — can still be defined and calculated.

As a further difficulty, processes obtained from cascades have positive increments $X(k)$, so that the argument (1.18) of Section 1.5 using the variances has to be corrected to read

$$2^{-2i}\, 2^{(i-n)(1+T(2))} \simeq \mathbb{E}\big[|X^{(m)}|^2\big] = \mathrm{var}(X^{(m)}) + \mathbb{E}\big[X^{(m)}\big]^2 \simeq \mathrm{var}(X) 2^{i(2H-2)} + \mathbb{E}[X]^2,$$

noting that $\mathbb{E}\big[X^{(m)}\big]$ is independent of the scale m. Since $2H-2<0$ we may, thus, still expect the same relation (7.23), at least in the limit of very fine resolution (small m and i).

The variance-time plot method (7.20) is known to be an unreliable (but simple) estimator of LRD behavior [99], while the wavelet method (7.22) is more robust. Recalling that $T_{w,X}(2)$ is defined in terms of wavelet energy, i.e., the variance of the wavelet coefficients of X, we find by stationarity $\mathrm{var}(C_{j,k}) = 2^{-j}\,\mathbb{E}[2^j|C_{j,k}|^2] \simeq 2^{-j}\,2^{-j(1+T(2))} = 2^{-j(2+T(2))}$, i.e.,

$$H_{\mathrm{wave}} = \frac{T_{w,X}(2)+3}{2}. \tag{7.24}$$

Example 7.1. (LRD parameter of Binomial Cascades) Together with (6.3) we find, e.g., for the binomial $Y=\mathcal{M}_b$:

$$H_{\mathrm{wave},\mathcal{M}_b} = \frac{T_{w,\mu_b}(2)+3}{2} = \frac{T_{\mathcal{M}_b}(2)+1}{2} = H_{\mathrm{var},\mathcal{M}_b}. \tag{7.25}$$

♠

This formula has interesting similarities with (6.2).

Example 7.2. (When scaling is only asymptotic) The examples presented so far in this section are processes with 'perfect' or 'omni-scaling' such as fBm and cascades, meaning that the scaling parameters H and $T(q)$ can be obtained from comparing statistics of any two scales. In real world applications one often requires more flexibility and matches the LRD, resp. multifractal parameters only in the limit of large, resp. small scales. Potentially, the two matching procedures can be performed independent of each other.

Linear processes such as FARIMA (or ARFIMA) [100] allow, e.g., to match short term correlations in addition to LRD. As a first step towards decoupling LRD and multifractal scaling we mention here cascades with scale dependent multipliers. Such processes were used in [89] to model internet traffic. In doing so we also demonstrate how misleading *time-domain* estimators such as $(T(2)+1)/2$ can be when only few scales of resolutions are available and a large mean 'masks' the fine scale variability. Let us add that 'fBm in multifractal time' which we introduce in the remaining section provides an attractive alternative to model LRD separately from fine scale properties.

Consider a binomial cascade supported on $[0,1]$ with (i) and (ii), but where instead of (iii) the distributions of $M_k^{(n)}$ depend on the scale n as follows:

$$\mathrm{var}(M_k^{(n)}) = \frac{2^{2-2H}\mathrm{var}(M_k^{(n-1)})}{4\mathrm{var}(M_k^{(n-1)})+1}. \tag{7.26}$$

This is arranged such that the Haar wavelet coefficients scale *exactly* as in (7.21). This powerlaw can be easily derived using (5.13) (see below for a rough argument or see [89]). In short, it is fair to say that this process has a scaling of the LRD type with exponent H.

To fully determine the distributions assume that $M_k^{(n-1)}$ follows a symmetrical β-distribution on $[0,1]$ (compare Example 5.1 or [89]). To start the iteration, we assume a value of $\mathbb{E}[(M_k^{(1)})^2] \simeq 1/4$ (but different from $1/4$) or $\mathrm{var}(M_k^{(1)}) \simeq 0$. This reflects typical findings with high resolution network traces where there is often little variation in the data on high aggregation levels in the order of several minutes.

Iterating (7.26) we find for the first few n that $\mathbb{E}[(M_k^{(n)})^2] \simeq 1/4$ and $\mathrm{var}(M_k^{(n)}) \simeq \sigma^2(2^{2-2H})^n$, which is small for small n. Only for very fine scales (large n) will $\mathbb{E}[(M_k^{(n)})^2] = \mathrm{var}(M_k^{(1)}) + 1/4$ converge to the limiting 2^{-2H}. All this will affect the estimation of the LRD parameter through $(T(2)+1)/2$ from limited data as we explain now.

Motivated by (5.24) we use $-(1/n)\log_2(2^n \prod_{i=1}^n \mathbb{E}[(M_k^{(i)})^2])$ as an estimator for $T(2)$. If measurements are available only for small n this will result in $T(2) \simeq -(1/n)\log_2(2^n 4^{-n}) = 1$ since convergence is slow, and $(T(2)+1)/2 \simeq 1$. Only in the limit $n \to \infty$ will one see $T(2) \simeq -(1/n)\log_2(2^n 2^{-2Hn}) = 2H-1$ and discover the 'correct' value $(T(2)+1)/2 \simeq H$.

This is in stark contrast to the ability of the wavelet based estimator H_{wave} to pick up H even from observing only few scales (recall that (7.21) holds exactly). Indeed, we argue that (7.21) should hold even when n is still small (compare [89] for a rigorous proof of the exact powerlaw). According to (5.13) and using $M_{2k_n}^{(n+1)} - M_{2k_n+1}^{(n+1)} = 2M_{2k_n}^{(n+1)} - 1$ we may translate the above approximations into

$$\mathbb{E}[C_{n,k}^2] = 2^n \mathbb{E}[(M_k^{(1)})^2] \cdots \mathbb{E}[(M_k^{(n)})^2] \cdot 4\mathrm{var}(M_k^{(n+1)}) \simeq 2^n 4^{-n} 4 (2^{2-2H})^n = 4(2^{1-2H})^n \tag{7.27}$$

as claimed. Consequently, the wavelet energy reveals $H_{\mathrm{wave}} = H$ even for small scales.

We see the reason for this phenomenon in the fact that $T(2)$ is based on non-centered 2^{nd} moments: for this estimator of LRD the scaling behavior is at coarser scales hidden behind the mean of the process. In further support of this conclusion we mention that also $H_{\mathrm{var}} \simeq H$ even when estimated from few resolutions (n small): indeed, the 'first order' approximation $\mathbb{E}[(M_k^{(i)})^2] \simeq 1/4$ ($i = 1, \ldots, n-1$), $\mathbb{E}[(M_k^{(n)})^2] = 1/4 + \mathrm{var}(M_k^{(n)})$ results in $\mathrm{var}(X^{(2^{-n})}) = \mathbb{E}[(M_k^{(1)})^2] \cdots \mathbb{E}[(M_k^{(n)})^2] - (1/2^n)^2 \simeq 1/4^{n-1}\mathrm{var}(M_k^{(n)}) \simeq \sigma^2/4 \cdot 2^{-2Hn}$, which is the claimed scaling.

As a final remark we add that with the 'initial condition' $\mathbb{E}[(M_k^{(1)})^2] = 2^{-2H}$ scaling is perfect and (7.25) holds again. ♠

8 Multifractal Time Change

So far, we have given examples of processes which are either statistically self-similar and 'mono-fractal' or multiplicative and almost surely increasing‖ and multifractal. It is clear that there is need to marry the two in order to have processes with richer structure. To quote Mandelbrot the idea towards this goal is "simple but inevitable

‖ Our multipliers are positive. Vector-valued multipliers can be treated but lead to technical difficulties [31].

after the fact": *multifractal time change.*

Indeed, 'Brownian motion in multifractal time' appeals through the simplicity in which it combines attractive features such as Gaussianity, LRD and multifractal structure. Arguing that trading occurs in 'multifractal time', [68, 69] introduced this process as a model for stock exchange. Here, we are exploring all its multifractal spectra, as well as more general 'multifractal subordinations'. Since most of our analysis can be done pathwise, let us start with a deterministic example. Recall, that we denote the spectra of a process Y computed with singularity exponents $s_k^{(n)}$ by $f_{s,Y}$ etc. when confusion is possible.

8.1 Weierstrass Function in Multifractal Time

The Weierstrass function

$$W(t) := \sum \lambda^{-kH} \sin(\lambda^k t) \tag{8.1}$$

has originally been introduced as an example of a nowhere differentiable function. In its randomized version it was proposed by Mandelbrot as an efficient and simple model of fBm. Similarly to fBm, $W(t)$ is a mono-fractal, more precisely, its local Hölder exponent $h_W(t)$ is everywhere equal to H. More pointedly, there are c and c' such that

$$|W(t+\delta) - W(t)| \le c\delta^H \qquad |W(t+\varepsilon) - W(t)| \ge c'\varepsilon^H \tag{8.2}$$

for all t, all δ and some $\varepsilon < \delta$.

In order to leave uniform Hölder continuity behind and moving towards multifractal scaling we proceed as follows:

Definition 8.1. *Let $\mathcal{M}(t) = \mu([0,t])$ be a multifractal distribution function. Then, the* Weierstrass function in multifractal time *W(MF) is given by*

$$\mathcal{W}(t) := W(\mathcal{M}(t)) \tag{8.3}$$

We claim that W(MF) has a significant multifractal structure. Indeed, assuming that $\mathcal{M}$ is continuous in a neighborhood of t we find a $\delta_0(t) > 0$ such that for $\delta < \delta_0(t)$

$$\begin{aligned} \sup_{[t-\delta,t+\delta]} |\mathcal{W}(s') - \mathcal{W}(s)| &= \\ \sup\{\, |W(u') - W(u)| &: \inf_{[t-\delta,t+\delta]} \mathcal{M}(s) \le u < u' \le \sup_{[t-\delta,t+\delta]} \mathcal{M}(s) \,\} \\ &\le c \sup_{[t-\delta,t+\delta]} |\mathcal{M}(s') - \mathcal{M}(s)|^H \end{aligned} \tag{8.4}$$

and the reverse inequality with c'. From this, it follows immediately that for all $\varepsilon > 0$ and $n > n_0(\varepsilon, t)$

$$|h_{k_n}^{(n)}(\mathcal{W}) - H \cdot h_{k_n}^{(n)}(\mathcal{M})| < \varepsilon. \tag{8.5}$$

This yields

Lemma 8.2 (Warp formula for Weierstrass). *Assume that $\mathcal{M}$ is piecewise continuous on $[0,1]$ with at most a finite number of discontinuities. Then, in points of continuity of $\mathcal{M}$*

$$h_{\mathcal{W}}(t) = H \cdot h_{\mathcal{M}}(t) \tag{8.6}$$

and

$$\dim(E^{[a]}_{h,\mathcal{W}}) = \dim(E^{[a/H]}_{h,\mathcal{M}}). \tag{8.7}$$

Moreover, if $\mathcal{M}$ is continuous, then it is actually uniformly continuous on $[0,1]$ and it is easy to see that δ_0 and n_0 of (8.4) are independent of t. Thus, we may relate the grain based spectra of $\mathcal{W}$ and $\mathcal{M}$ using (8.5), i.e., $f_{\mathcal{W}}(a) = f_{\mathcal{M}}(a/H)$. Their partition functions are connected by a similar formula as we are about to develop. Referring back to (8.4) we find, e.g., for positive q, and $n \geq n_0$

$$S^{(n)}_{h,\mathcal{W}}(q) \;=\; \sum_{k=0}^{2^n-1} 2^{-qnh^{(n)}_k(\mathcal{W})} \leq c^{qH} \sum_{k=0}^{2^n-1} 2^{-qnH\cdot h^{(n)}_k(\mathcal{M})} = c^{qH} S^{(n)}_{h,\mathcal{M}}(qH).$$

Similar estimates using the reverse of (8.4) show that $S^{(n)}_{h,\mathcal{W}}(q)$ and $S^{(n)}_{h,\mathcal{W}}(q)$ are equal up to a factor which is bounded independently of n. So, if $\tau_{\mathcal{M}}(qH)$ assumes a limit as in (3.12), then so does $\tau_{\mathcal{W}}(q)$. In summary

Lemma 8.3. *Assume that $\mathcal{M}$ is continuous on $[0,1]$. Then,*

$$\tau_{h,\mathcal{W}}(q) = \tau_{h,\mathcal{M}}(qH) \qquad f_{h,\mathcal{W}}(a) = f_{h,\mathcal{M}}(a/H). \tag{8.8}$$

These lemmas say nothing more than what was already apparent in (8.2): the Weierstrass function stretches distances locally in a very uniform way in the form of a powerlaw with exponent H. In W(MF) in particular, it stretches the multifractal time increments, multiplying local Hölder exponents by H. As a consequence we have:

Theorem 8.4 (Multifractal formalism for Weierstrass). *Let $\mathcal{W}$ be defined as in (8.3) and assume that $\mathcal{M}$ is continuous. Consider the singularity exponents $h^{(n)}_k$. Then, for every a with*

$$\dim(E^{[a/H]}_{\mathcal{M}}) = \tau^*_{\mathcal{M}}(a/H)$$

equality holds in the **multifractal formalism***:*

$$\dim(E^{[a]}_{\mathcal{W}}) = f_{\mathcal{W}}(a) = \tau^*_{\mathcal{W}}(a) = \tau^*_{\mathcal{M}}(a/H). \tag{8.9}$$

More delicate questions of multifractal analysis find again a simple answer provided $\mathcal{M}$ is continuous. E.g., if $E^{[a/H]}_{\mathcal{M}}$ has positive finite γ-dimensional Hausdorff measure then $E^{[a]}_{\mathcal{W}}$ has positive finite γ-dimensional Hausdorff measure where $\gamma = \dim(E^{[a/H]}_{\mathcal{M}}) = \dim(E^{[a]}_{\mathcal{W}})$, etc. Similarly, it follows that $\mathcal{W}$ has a rich multifractal structure if $\mathcal{M}$ has one. By this we mean that the sets $E^{[a]}_{\mathcal{W}}$ are dense on the unit interval. A smooth, strictly concave partition function $\tau_{\mathcal{W}}$ can be taken as an indicator of highly interwoven sets $E^{[a]}$.

The particular form of (8.9) and (8.8) allows to separate the Hölder continuity H of the Weierstrass function from the multifractal spectrum of the 'time change' $\mathcal{M}$, even numerically. Indeed, the slope of the tangent of the $\mathcal{W}$-spectra through the origin have slope $1/H$ since $\tau_{\mathcal{W}}(1/H) = \tau_{\mathcal{M}}(1) = 0$. Alternatively, one may look for the zero of $\tau_{\mathcal{W}}$ in order to determine H. Once this is accomplished $\tau_{\mathcal{M}}$ follows easily.

8.2 Self-similar processes in multifractal time

Following the example of the last section it is now easy to define processes with rich, ubiquitous multifractal structure, yet attractive statistical properties such as zero mean marginals and second order stationarity, simply by *multifractal time change*. In order to be able to compute multifractal spectra, one needs some scaling properties of the subordinated process, of course. To this end, let V denote a H-sssi process and $\mathcal{M}(t)$ an almost surely increasing process from which we assume to know certain of its multifractal spectra. Then, set

$$\mathcal{V} := V(\mathcal{M}(t)). \tag{8.10}$$

As always, the deterministic partition function $T(q)$ is the most simple of the spectra to calculate. Indeed, the computation for H-sssi processes carried out in the introduction (see (1.24)) can easily be generalized.

Theorem 8.5. *Let V denote an H-sssi process and $\mathcal{M}$ an independent time change. Let $\mathcal{V} := V(\mathcal{M}(\cdot))$. Then, with $\underline{q}$ and $\overline{q}$ as defined in lemma 7.4,*

$$T_{\mathcal{V}}(q) = \begin{cases} T_{\mathcal{M}}(qH) & \text{if } \underline{q} < q < \overline{q} \\ -\infty & \text{else.} \end{cases} \tag{8.11}$$

Here, the singularity exponents may be chosen to be $\alpha_k^{(n)}$ without further assumption. For $h_k^{(n)}$ the result holds provided $\mathcal{M}$ has almost surely continuous paths. Finally, in the case of a binomial warp time $\mathcal{M}_b$ as in Section 5, (8.11) holds also for $w_k^{(n)}$ for any admissible, analyzing wavelet supported on $[0, 1]$ under some technical condition.

Note that $T_{\mathcal{V}}(q)$ may assume the value $-\infty$ even for $\underline{q} < q < \overline{q}$ depending on whether $T_{\mathcal{M}}(qH)$ is finite or not.

Proof

First, assuming continuous warp time $\mathcal{M}$ we may deal with suprema as in (8.4). This takes care of the case $h_k^{(n)}$.

Let us now extend (1.24) to $w_k^{(n)}$. When conditioning on $\mathcal{M}$ we obtain by H-sssi and (5.16)

$$\mathcal{V}(t) - \mathcal{V}(k_n 2^{-n})\big|_{\mathcal{M}} \stackrel{\text{fd}}{=} \Big(M_{k_n}^{(n)} \cdots M_{k_1}^{(1)} \cdot \mathcal{M}_b^{(n,k_n)}(2^n t - k_n) \Big)^H .$$

The admissibility of the wavelet ($\int \psi = 0$) allows now to write

$$
\begin{aligned}
2^{n/2} C_{n,k_n}(\mathcal{V})\,\big|_{\mathcal{M}} &= 2^n \int_{I_{k_n}^{(n)}} \psi(2^n t - k_n)\mathcal{V}(t)\mathrm{d}t\,\big|_{\mathcal{M}} \\
&= 2^n \int_{I_{k_n}^{(n)}} \psi(2^n t - k_n)\left(\mathcal{V}(t) - \mathcal{V}(k_n 2^{-n})\right)\mathrm{d}t\,\big|_{\mathcal{M}} \\
&\overset{\mathrm{d}}{=} \left(M_{k_n}^{(n)} \cdots M_{k_1}^{(1)}\right)^H A_{n,k_n}\left(\mathcal{M}_b^H\right) \cdot V(1).
\end{aligned}
\tag{8.12}
$$

Hereby, $A_{n,k_n}(\mathcal{M}_b^H)$ stands for the integral

$$
\int_0^1 \psi(t)(\mathcal{M}_b(t))^H \mathrm{d}t = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \psi(k2^{-n})(\mathcal{M}_b(k2^{-n}))^H \cdot 2^{-n}
\tag{8.13}
$$

and is, actually, nothing but the wavelet coefficient $C_{0,0}$ of $(\mathcal{M}_b^{(n,k_n)})^H$. Since $\mathcal{M}_b^{(n,k_n)} \overset{\mathrm{d}}{=} \mathcal{M}_b$, we have $\mathbb{E}_\Omega[|A_{n,k_n}|^q] = \mathbb{E}_\Omega[|C_{0,0}(\mathcal{M}_b^H)|^q]$ and the integral term will not affect the scaling of moments, provided $\mathbb{E}_\Omega[|C_{0,0}(\mathcal{M}_b^H)|^q]$ is finite. This is one part of our technical assumption.

Similarly, $\mathbb{E}_\Omega[|V(1)^q]$ will not affect $T_{w,\mathcal{V}}$ as long as it is finite. This is taken into account by restricting q to $(\underline{q}, \overline{q})$.

Comparing with the derivation (5.24) one finds now easily that $T_{w,\mathcal{V}}(q) = T_{\alpha,\mathcal{M}_b}(qH)$. Assuming now also that $\mathbb{E}_\Omega[|C_{0,0}(\mathcal{M}_b)|^{qH}]$ is finite, which is the second part of the technical assumption, then $T_{\alpha,\mathcal{M}_b}(qH) = T_{w,\mathcal{M}_b}(qH)$ by (6.3). So, (8.11) is established. ♢

8.3 FBm in multifractal time

The most obvious example of a 'warped motion' is *fractional Brownian motion in multifractal time* FB(MF), i.e., $\mathcal{B} = B_H(\mathcal{M})$. It is first mentioned in [68, 69] which pioneered it's application to stock markets. FB(MF) has particularly nice properties. It must be distinguished from the 'multifractional Brownian motion' of [85] which has a 'slowly varying' $h(t)$, non-ubiquitous singularities, a piecewise linear T^*, and strict inequality in the multifractal formalism [62], i.e., an excellent model for non-stationarity rather than for multifractality.

Pathwise Spectra of warped fBm

For any process V which has strict and uniformly Hölder continuous paths such as fBm, the pathwise spectra are as easy to compute as for the multifractal Weierstrass function. Indeed, applying Adler's result (see Theorem 7.3) for $\eta = 1/m$ ($m \in \mathbb{N}$) instead of (8.2) we find that with probability one, for all $m \in \mathbb{N}$ there is a finite, positive A_m and $n_0(m) \in \mathbb{N}$ such that

$$
-\frac{1}{n}\log_2(A_m)(H - 1/m)h_{k_n}^{(n)}(\mathcal{M}) \le h_{k_n}^{(n)}(\mathcal{B}) \le \frac{1}{n}\log_2(A_m)(H + 1/m)h_{k_n}^{(n)}(\mathcal{M}).
\tag{8.14}
$$

for all t and all $n > n_1(m,t)$. As usual, $h^{(n)}_{k_n} = k_n(t) = \lfloor t2^n \rfloor$ as in definition 5.1. Here, n_1 has to be taken large enough such that $\mathcal{M}$ is continuous in $[(k_n-1)2^{-n},(k_n+2)2^{-n}]$ (compare (8.4)) and such that $|\mathcal{M}(s') - \mathcal{M}(s)| \leq 2^{-n_0}$ for all $s, s' \in [(k_n-1)2^{-n},(k_n+2)2^{-n}]$. To avoid complications we should assume here that time change $\mathcal{M}$ and fBm B_H are independent.

Assuming that $\mathcal{M}$ has almost surely continuous paths this allows to conclude immediately that with probability one,

$$h_{\mathcal{B}}(t) = H \cdot h_{\mathcal{M}}(t)$$

for all t in a compact interval. When $\mathcal{M}$ is continuous, however, then n_1 can be chosen independently of t with probability one, and we may compare grain based spectra and the partition functions in the same way as for the warped Weierstrass function W(MF). So, lemma 8.3 and theorem 8.4 hold for warped fBm FB(MF) mutatis mutandis:

Theorem 8.6 (Warp formula for fBm). *Let B_H denote fBm, let $\mathcal{M}$ be an almost surely continuous time change independent of B_H and set $\mathcal{B}(t) := B_H(\mathcal{M}(t))$. Then, for almost every path (8.7) and (8.8) hold with $\mathcal{W}$ replaced by $\mathcal{B}$. In particular, for almost every path*

$$\dim(E^{[a]}_{\mathcal{B}}) = f_{\mathcal{B}}(a) = \tau^*_{\mathcal{B}}(a) = \tau^*_{\mathcal{M}}(a/H) \tag{8.15}$$

for all a with

$$\dim(E^{[a/H]}_{\mathcal{M}}) = \tau^*_{\mathcal{M}}(a/H),$$

where all these spectra are taken with respect to the Hölder exponents $h^{(n)}_k$.

Estimation through wavelets

While theorem 8.6 is an appealing theoretical result, it is quite unaccessible from a numerical point of view because suprema of continuous time processes are hard to estimate from a discrete sampling. It comes as a remedy that wavelets can be used for estimation purposes as we are going to develop in a special case now.

Therefore let us assume that the warp time is a binomial cascade $\mathcal{M}_b$ satisfying (i)-(iii) (see (5.5)). Let ψ be an admissible, analyzing wavelet, i.e., $\int \psi = 0$, supported on $[0,1]$. Exploiting the uniform Hölder continuity of fBm (see (8.14) or Theorem 7.3) and the strong scaling property of $\mathcal{M}_b$ (see (5.14)) we will lay open the multiplicative structure of the wavelet coefficients $C_{n,k_n}(\mathcal{B})$ in a result analogous to lemma 5.4. This is, of course, much stronger than (8.12) which dealt only with distributions:

$$\begin{aligned} 2^{n/2} C_{n,k_n}(\mathcal{B}) &= 2^n \int_{I_{k_n}^{(n)}} \psi(2^n t - k_n)\mathcal{B}(t)\mathrm{d}t \\ &= 2^n \int_{I_{k_n}^{(n)}} \psi(2^n t - k_n)\left(V(\mathcal{M}_b(t)) - V(\mathcal{M}_b(k_n 2^{-n}))\right)\mathrm{d}t \\ &\approx 2^n \int_{I_{k_n}^{(n)}} \psi(2^n t - k_n)\left(\mathcal{M}_b(t) - \mathcal{M}_b(k_n 2^{-n})\right)^{H\pm\eta}\mathrm{d}t \\ &= \left(M_{k_n}^{(n)} \cdots M_{k_1}^{(1)}\right)^{H\pm\eta} \int_0^1 \psi(t')\left(\mathcal{M}_b^{(n,k_n)}(t')\right)^{H\pm\eta}\mathrm{d}t' \\ &= \left(2^{n/2} C_{n,k_n}(\mathcal{M}_b)\right)^{H\pm\eta} \frac{A_{n,k_n}\left(\mathcal{M}_b^{H\pm\eta}\right)}{\left(A_{n,k_n}(\mathcal{M}_b)\right)^{H\pm\eta}}. \end{aligned}$$

Hereby, we used $\approx$ to indicate equality up to a bounded error, i.e., A and K in the notation of theorem 7.3. Also, $A_{n,k_n}(\mathcal{M}_b^{H'})$ is as in (8.13) and we assume it to be uniformly bounded away from zero for $H' = 1$ as well as all H' close to H. Taking $\log_2$ and dividing by $-n$ we find the analogue to (8.14), i.e.,

$$w_{k_n}^{(n)}(\mathcal{B}) \approx H \cdot w_{k_n}^{(n)}(\mathcal{M}_b)$$

up to an error which will become uniformly small as n increases. In short, $w_{\mathcal{B}}(t) = H \cdot w_{\mathcal{M}_b}(t)$, $f_{w,\mathcal{B}}(a) = f_{w,\mathcal{M}_b}(a/H)$ etc. Since for a binomial the wavelet based spectra coincide with the ones based on $\alpha_k^{(n)}$ and $h_k^{(n)}$ almost surely we can summarize as follows:

Theorem 8.7 (Wavelet analysis of warped fBm). *Let $\mathcal{M}_b$ be a binomial time change satisfying (i)-(iii) (see (5.5)), independent of B_H. Consider an admissible, compactly supported analyzing wavelet. Assume that $A_{n,k}(\mathcal{M}_b^{H'})$ are uniformly bounded away from zero for $H' = 1$ and H' close to H. Then, for almost all paths*

$$\dim(E_{w,\mathcal{B}}^{[a]}) = \dim(E_{h,\mathcal{B}}^{[a]}) = f_{w,\mathcal{B}}(a) = f_{h,\mathcal{B}}(a) = \tau_{w,\mathcal{B}}^*(a) = \tau_{h,\mathcal{B}}^*(a) = T_{\mathcal{M}_b}^*(a/H) \quad (8.16)$$

for all a with $T_{\mathcal{M}_b}^*(a) > 0$.

This is impressively demonstrated in Figure 3. For the statistics of the estimator for $\tau_{w,\mathcal{B}}^*(a)$ used in there see [40].

We would like to point out that this result is not so much due to the *dyadic* structure of both the binomial warp time $\mathcal{M}_b$ as well as the wavelet coefficients $C_{n,k}$, but rather the strong *rescaling* properties of $\mathcal{M}_b$ and fBm. Using sophisticated tools which are now standard in multifractal analysis it is very well possible to extend this result to arbitrary *multi-nomial* warp time as introduced in Section 5.6.

LRD-parameter for warped fBm

An immediate consequence of (8.11) is a formula for the ‘LRD’ parameter of FB(MF). We could employ (7.20) or (7.22). However, we choose to calculate correlations exactly.

In order to do so, we develop $(\mathcal{B}(t) - \mathcal{B}(s))^2$ and take expectations as in (7.2). First note that

$$\mathbb{E}[\mathcal{B}(t)^2] = \mathbb{E}\mathbb{E}[B_H(\mathcal{M}(t))^2|\mathcal{M}(t)] = \mathbb{E}[\mathcal{M}(t)^{2H}] \cdot \mathbb{E}[B_H(1)^2].$$

Writing $\sigma^2 = \mathbb{E}[B_H(1)^2]$ for short we obtain

$$\mathbb{E}[\mathcal{B}(t)\mathcal{B}(s)] = \frac{\sigma^2}{2}\Big(\mathbb{E}[\mathcal{M}(t)^{2H}] + \mathbb{E}[\mathcal{M}(s)^{2H}] - \mathbb{E}[|\mathcal{M}(t) - \mathcal{M}(s)|^{2H}]\Big). \tag{8.17}$$

Let us consider now the increment process for a fixed lag size $\delta < 1$, i.e., fractional Gaussian noise in multifractal time FG(MF):

$$\mathcal{G}(s) := \mathcal{B}(s+\delta) - \mathcal{B}(s) = B_H(\mathcal{M}(s+\delta)) - B_H(\mathcal{M}(s)). \tag{8.18}$$

Writing $\Delta_a^b = \mathcal{M}(b) - \mathcal{M}(a)$ for short we find from (8.17)

$$\mathbb{E}\left[\,\mathcal{G}(s)\cdot\mathcal{G}(t)\,\right] = \frac{\sigma^2}{2}\mathbb{E}\left[|\Delta_t^{s+\delta}|^{2H} - |\Delta_{t+\delta}^{s+\delta}|^{2H} - |\Delta_t^{s}|^{2H} + |\Delta_{t+\delta}^{s}|^{2H}\right]. \tag{8.19}$$

Lemma 8.8 (LRD of warped fBm). *Assume that for each k the moment of order $2H$ of the increments $\mathcal{M}(k\delta) - \mathcal{M}(m\delta)$ depends only on $|k-m|$ and is finite. Then, the discrete time FG(MF) is second order stationary with auto-correlation function*

$$\begin{aligned} r_{\mathcal{G}}(k) &= \mathbb{E}_\Omega[\mathcal{G}(n+k)\mathcal{G}(n)] \\ &= \frac{\sigma^2}{2}\mathbb{E}_\Omega\left[\mathcal{M}((k+1)\delta)^{2H} - 2\mathcal{M}(k\delta)^{2H} + \mathcal{M}((k-1)\delta)^{2H}\right] \end{aligned} \tag{8.20}$$

The assumptions apply certainly if $\mathcal{M}$ is a Lévy subordinator, i.e., an H'-sssi $1/H'$-stable process, provided $2H < 1/H'$. Strictly speaking, a subordinator should be a.s. increasing which is the case for Lévy motion provided $H' > 1$. Note that we do not require monotonicity of $\mathcal{M}$. A binomial 'subordinator' satisfy the assumptions of the lemma with $H = 1/2$ since $\mathbb{E}_\Omega[\mathcal{M}_b(t)] = t$. Finally, we note that multiplicative multifractal processes with stationary increments have been introduced recently [70, 74, 105].

The formula (8.20) extends the usual auto-correlation function of fGn. Indeed, when choosing $\mathcal{M}(t) = t$ it reduces to (7.3), as it should. In general, considering (8.20), (7.20) and (7.22), it is probably fair to call

$$H_{\mathcal{G}} = \frac{T_{\alpha,\mathcal{M}}(2H) + 1}{2} \tag{8.21}$$

the LRD parameter of fractional Gaussian noise in multifractal time.

Consider the special case $H = 1/2$ and $\mathcal{M}$ increasing with $\mathbb{E}_\Omega[\mathcal{M}(t)] = t$. Then, $\mathcal{G}$ is simply 'white (i.i.d.) Gaussian noise in multifractal time'. Its (2^{nd} order) correlations vanish due to (8.20). This agrees with $H_{\mathcal{G}} = 1/2$ which holds due to $T_{\mathcal{M}}(1) = 0$. Nevertheless, the process $\mathcal{G}$ is highly dependent which is revealed by the higher order moments, i.e., by $T_{\mathcal{B}}(q)$ for $q \neq 2$. Mandelbrot calls this the 'blind spot' of spectral analysis.

Simulation

For simulations of fBm in multifractal time FB(MF) it is advisable to use a randomized Weierstrass function

$$\mathcal{W}(t) := \sum_{k=1}^{\infty} C_k \lambda^{kH} \sin(\lambda^k \mathcal{M}(t) + A_k).$$

Since $|\mathcal{M}(t)-\mathcal{M}(s)|$ is for most s and t much smaller than $|t-s|$ (actually $\simeq |t-s|^{\alpha(t)}$, with most $\alpha(t) \geq \alpha_0 > 1$) any other procedure would require high precision simulation of fBm and might be numerically demanding. With C_k i.i.d. $\mathcal{N}(0,1)$ and A_k i.i.d. $\mathcal{U}[0,2\pi)$, $\mathcal{W}$ has zero mean increments with variance (compare [32, p 248])

$$\begin{aligned}
\mathbb{E}[(\mathcal{W}(t+\delta) - \mathcal{W}(t))^2 | \mathcal{M}] &= 2\sum_{k=1}^{\infty} \lambda^{2kH} \sin^2(\lambda^k(\mathcal{M}(t+\delta) - \mathcal{M}(t))) \\
&\simeq c(\mathcal{M}(t+\delta) - \mathcal{M}(t))^{2H}.
\end{aligned}$$

This gives hope that at least the correlation decay is matched.

Another simple case for simulation is Wiener motion in multifractal time WM(MF) because WM, i.e., $B_{1/2}$, has independent increments (compare Figure 3 (a)-(c)).

8.4 Pathwise spectra for warped H-sssi processes

In this section we provide two types of formulas for the pathwise spectra similar as before: the 'multifractal formalism' relates various spectra of subordinated processes with each other, while the 'warp formulas' relates them to the spectra of the subordinator.

A first step towards this goal is to compute the deterministic grained spectrum $F(a)$. This is somewhat more involved than dealing with $T(q)$, though we may follow the computation carried out for simple H-sssi processes (compare (7.11)) with only little modification. The effort is worthwhile since we will gain information on the pathwise spectra $f(a)$.

The strong monofractal behavior of fBm relates the spectra of the time change $\mathcal{M}$ and the 'warped' process V in an almost trivial manner (see theorem 8.6) and provides us with a simple intuition: scaling exponents multiply under subordination. Since the exponent of fBm is constant, i.e., $h(t) = H$ for all t, it is easy to control the warped spectra. Allowing general H-sssi processes, however, brings about the problem that many different combinations of singularity exponents of $\mathcal{M}$ and V may multiply to yield the same exponent for $\mathcal{V} = V(\mathcal{M}(\cdot))$. This will generally result in 'smoother' spectra.

Formulas and discussion

From the previous discussion we may expect that the spectra of the warped process $\mathcal{V}$ and the subordinator $\mathcal{M}$ are related through simple transforms which account for the aforementioned 'smoothing'. We first introduce and discuss these transforms on the spectra and establish then conditions under which they equal the spectra of the warped process. For simplicity we stick in this section with the case of subordinating self-similar processes, leaving the general processes for Section 8.5 when the necessary formulas appear naturally.

Definition 8.9. *The* H-warped partition functions of $\mathcal{M}$ *are defined as*

$$\text{(}H\text{-sssi:)} \qquad T^{\|}_{\mathcal{M}}(q) := \begin{cases} T_{\mathcal{M}}(qH) & \text{for } \underline{q} < q < \overline{q}, \\ -\infty & \text{else,} \end{cases} \tag{8.22}$$

and similarly for $\tau^{\|}_{\mathcal{M}}(q)$*. Note that* $T^{\|}_{\mathcal{M}}(q)$ *may be* $-\infty$ *even for* $\underline{q} < q < \overline{q}$*.*

With the symbol $^{\|}$ we try to indicate the truncation of the functions. In this notation, (8.11) becomes simply $T_{\mathcal{V}}(q) = T^{\|}_{\mathcal{M}}(q)$, expressing that the operation $(\cdot)^{\|}$ gives the correct passage from $T_{\mathcal{M}}$ to $T_{\mathcal{V}}$.

Taking the Legendre transform of $\tau^{\|}_{\mathcal{M}}$, e.g., we find

$$(\tau^{\|}_{\mathcal{M}})^*(a) = \inf_q \left(qa - \tau^{\|}_{\mathcal{M}}(q)\right) = \inf_{\underline{q}<q<\overline{q}} (qa - \tau_{\mathcal{M}}(qH)) = \inf_{H\underline{q}<q'<H\overline{q}} \left(q'\frac{a}{H} - \tau_{\mathcal{M}}(q')\right). \tag{8.23}$$

From the last of these expressions we learn that $(\tau^{\|}_{\mathcal{M}})^*(a) = \tau^*_{\mathcal{M}}(a/H)$ whenever $\underline{q} = -\infty$ and $\overline{q} = \infty$. This is the case for Wiener motion $B_{1/2}$ when choosing the Hölder exponents $h^{(n)}_k$. So, theorem 8.6 tells us that $(\tau^{\|}_{\mathcal{M}})^*(a)$ could indeed be a good guess at $f_{\mathcal{V}}$ for general warped H-sssi processes, but obviously only where $f_{\mathcal{V}}$ is concave. This leads us to the following definition, still in the context of subordinated self-similar processes (for general processes see definition 8.14):

Definition 8.10. *Assume that* $f_{\mathcal{M}}$*, respectively* $F_{\mathcal{M}}$*, are* concave*. Then, the* H-warped spectra of $\mathcal{M}$ *are given by*

$$\text{(}H\text{-sssi, concave):} \qquad f^{\triangle}_{\mathcal{M}}(a) := (\tau^{\|}_{\mathcal{M}})^*(a), \qquad F^{\triangle}_{\mathcal{M}}(a) := (T^{\|}_{\mathcal{M}})^*(a). \tag{8.24}$$

With the symbol $^{\triangle}$ we try to indicate the transformed shape of the functions (compare Figure 9 (b)) as summarized in the following:

Lemma 8.11. *Assume that* $f_{\mathcal{M}}$*, respectively* $F_{\mathcal{M}}$*, are* concave*. Let* $f^{\triangle}$ *and* $F^{\triangle}$ *be given by (8.24).*
(a) Shape: *For* $a = H\tau'_{\mathcal{M}}(qH)$ *with* $\underline{q} < q < \overline{q}$

$$f^{\triangle}_{\mathcal{M}}(a) = f_{\mathcal{M}}(a/H) \qquad \text{and} \qquad (f^{\triangle}_{\mathcal{M}})'(a) = q. \tag{8.25}$$

For smaller, resp. larger a *the function* $f^{\triangle}_{\mathcal{M}}(a)$ *is linear with slopes* $\overline{q}$ *and* $\underline{q}$*, resp.*
(b) Warping of the Multifractal formalism: *If* $\tau_{\mathcal{M}}(qH) = T_{\mathcal{M}}(qH)$ *for all* $\underline{q} < q < \overline{q}$*, then for all* a

$$f^{\triangle}_{\mathcal{M}}(a) = \left(\tau^{\|}_{\mathcal{M}}\right)^*(a) = \left(T^{\|}_{\mathcal{M}}\right)^*(a) = F^{\triangle}_{\mathcal{M}}(a) \tag{8.26}$$

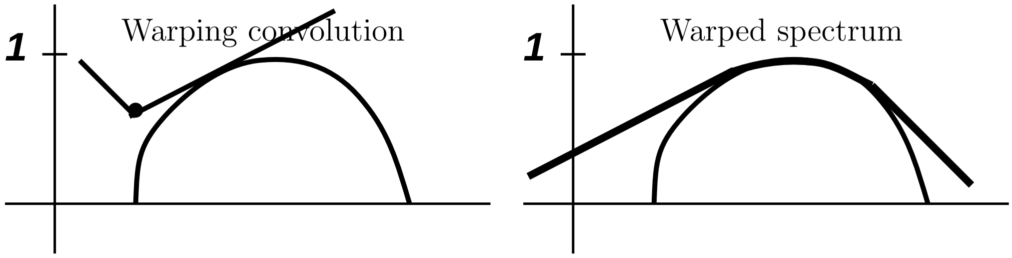

FIGURE 9. *Left: To 'warp' a concave function f with a piecewise linear function such as the grain spectrum $g = F_V$ of a stable Lévy motion one considers the 'morphed' curves $g_{y,z}(x) = z - x(g(y/x) - 1)$, singling out the ones which touch f. Right: The resulting warped function $f^{\triangle}(H\cdot)$ is the set of minima of those touching $g_{y,z}$. (Compare 8.4.)*

Proof
Claim (a) follows from (8.23). Claim (b) follows from (a) using corollaries 4.5 and 4.7. ♢

The following theorem establishes in what sense the warped spectra of the subordinator $\mathcal{M}$ equal the spectra of the warped motion. The proof shows that the theorem holds also for non-concave spectra, thereby extending $f^{\triangle}$ to the general case.

Theorem 8.12 (Self-similar processes in multifractal time). *Let V be an H-sssi process and $\mathcal{M}$ an arbitrary time change,* independent *of V. Denote $\mathcal{V} := V(\mathcal{M}(\cdot))$. Admit the singularity exponents $\alpha_k^{(n)}$, $h_k^{(n)}$ and $w_k^{(n)}$ under the same conditions as in theorem 8.5. Then, conditioned on knowing $\mathcal{M}$***

$$F_{\mathcal{V}}(a)\,\big|_{\mathcal{M}} = f_{\mathcal{M}}^{\triangle}(a). \tag{8.27}$$

Furthermore,

$$F_{\mathcal{V}}(a) = F_{\mathcal{M}}^{\triangle}(a). \tag{8.28}$$

Finally, if $f_{\mathcal{M}} = \underline{f}_{\mathcal{M}}$ then $F_{\mathcal{V}}\big|_{\mathcal{M}} = \underline{F}_{\mathcal{V}}\big|_{\mathcal{M}}$ (compare (3.28) and (3.26)).

In the special case when V has independent increments theorem 3.14 allows to compute the pathwise grain spectra as well:

Corollary 8.13 (Lévy stable motion in multifractal time). *Let L_H denote Lévy stable motion and let $\mathcal{M}$ be an almost surely strictly increasing time change,* independent *of L_H. Set $\mathcal{V} := L_H(\mathcal{M}(\cdot))$. Admit the singularity exponents $\alpha_k^{(n)}$, $h_k^{(n)}$ and $w_k^{(n)}$ under the same conditions as in theorem 8.5. Assume that $f_{\mathcal{M}} = \underline{f}_{\mathcal{M}}$ almost surely. Let a be such that $f_{\mathcal{M}}^{\triangle}(a) > 0$ almost surely. Then, for almost all paths we have the* **warp formula**

$$f_{\mathcal{V}}(a,\omega) = F_{\mathcal{V}}(a)\,\big|_{\mathcal{M}} = f_{\mathcal{M}}^{\triangle}(a). \tag{8.29}$$

Moreover, the **multifractal formalism** *holds in the sense that*

$$f_{\mathcal{V}}(a,\omega) \overset{\text{a.s.}}{=} \tau_{\mathcal{V}}^*(a,\omega) \overset{\text{a.s.}}{=} F_{\mathcal{V}}(a) = T_{\mathcal{V}}^*(a) = (T_{\mathcal{M}}^{\|})^*(a) \tag{8.30}$$

**In abuse of notation we let $F_{\mathcal{V}}(a)\,\big|_{\mathcal{M}} = \lim_{\varepsilon} \limsup_n (1/n) \log_2 \mathbb{E}[N^{(n)}(a,\varepsilon)\,\big|\mathcal{M}]$.

for any a for which $\left(T^{\|}_{\mathcal{M}}\right)^*(a) \overset{\text{a.s.}}{=} \underline{f}^{\triangle}_{\mathcal{M}}(a) > 0.$

Let us discuss some examples and special cases.

Example 8.1. (Binomial subordinator) For a binomial time change $\mathcal{M}_b$ our results are the strongest, since all spectra, i.e., all dimension and grain based spectra of $\mathcal{M}_b$ for $\alpha_k^{(n)}$, $h_k^{(n)}$ as well as $w_k^{(n)}$, coincide and are strictly concave almost surely. In particular, $f_{\mathcal{M}_b}$ is concave, $\underline{f}_{\mathcal{M}_b} = f_{\mathcal{M}_b}$ and $\tau^{\|}_{\mathcal{M}_b} = T^{\|}_{\mathcal{M}_b}$. Consequently, $\left(T^{\|}_{\mathcal{M}_b}\right)^*(a) \overset{\text{a.s.}}{=} \underline{f}^{\triangle}_{\mathcal{M}_b}(a)$ and (8.30) of corollary 8.13 applies to the entire positive part of $\left(T^{\|}_{\mathcal{M}_b}\right)^*(a)$. ♠

Example 8.2. (Lévy subordinator) Let us assume that the warp time is Lévy stable motion with Hurst parameter H' while the subordinated process is Lévy stable motion with Hurst parameter H, i.e., $\mathcal{V} = L_H(L_{H'})$. Then, $\underline{f}_{L_{H'}} = f_{L_{H'}} = \tau^*_{L_{H'}} = T^*_{L_{H'}}$ by (7.17) and we may apply corollary 8.13. According to (7.13) these spectra are piecewise linear.

To compute $f^{\triangle}_{L_{H'}}$ let us consider for simplicity the singularity exponent $\alpha_k^{(n)}$. In this case $\overline{q} = 1/H$ and $\underline{q} = -1$. Using that $\tau_{L_{H'}}(qH) = qHH' - 1$ for $-1/H < q < 1/(HH')$ —again by (7.17) and (7.13)— it is straightforward to compute via (8.24) and (8.29)

$$f_{L_H(L_{H'})}(a) = (\tau^{\|}_{L_{H'}})^*(a) = \begin{cases} 1 + (a - H\cdot H')\min(\frac{1}{H}, \frac{1}{H\cdot H'}) & \text{if } a < H\cdot H' \\ 1 & \text{if } a = H\cdot H' \\ 1 - (a - H\cdot H')\min(1, \frac{1}{H}) & \text{if } a > H\cdot H'. \end{cases} \tag{8.31}$$

We will revisit this example in the next section and comment on how to understand this formula. ♠

Proof of theorem 8.12

As the reader will notice the only assumptions on V used in the proof are stationary increments and $F_V = \underline{F}_V$. We formulate the proof for $\alpha_k^{(n)}$ and $h_k^{(n)}$. Since only distributions matter here, (8.12) is sufficient to deal with the case of wavelet coefficients.

Let us compute first the relevant probabilities conditioned on $\mathcal{M}$. For the ease of notation we assume that $\mathcal{M}$ is monotonous so that $\inf_{I_k^{(n)}} \mathcal{M}(t) = \mathcal{M}(k2^{-n})$ and $\sup_{I_k^{(n)}} \mathcal{M}(t) = \mathcal{M}((k+1)2^{-n})$. This simplifies the formulas considerably.

We start by looking at

$$p_n(k, a') := P_\Omega[a - \varepsilon < s_k^{(n)}(\mathcal{V}) < a + \varepsilon \,\big|\, \mathcal{M},\, a' < s_k^{(n)}(\mathcal{M}) < a' + \varepsilon'] \tag{8.32}$$

Conditioning on knowing $\mathcal{M}$ we actually know $s_k^{(n)}(\mathcal{M})$ (and not only $a' < s_k^{(n)}(\mathcal{M}) < a' + \varepsilon'$). Due to stationarity of increments of V we have $p_n(k, a') = p_n(0, a')$. For $s_k^{(n)} = \alpha_k^{(n)}$ we find $s_0^{(n)}(\mathcal{V}) = |V(\,\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})\,)| = |V(2^{-ns_k^{(n)}(\mathcal{M})})|$. For $s_k^{(n)} = h_k^{(n)}$, $s_0^{(n)}(\mathcal{V})$ becomes the supremum of increments of V over an interval of length $2^{-ns_k^{(n)}(\mathcal{M})}$ adding only notational but no logical difficulties. Having expressed $s_k^{(n)}(\mathcal{V})$ through $s_k^{(n)}(\mathcal{M})$ we may now abbreviate $s_k^{(n)} = s_k^{(n)}(\mathcal{M})$ to simplify formulas without creating confusion. A simple manipulation yields

$$p_n(k,a') = p_n(0,a') = \tag{8.33}$$
$$P_\Omega\left[\left(2^{-ns_k^{(n)}}\right)^{(a+\varepsilon)/s_k^{(n)}} < |V(2^{-ns_k^{(n)}})| < \left(2^{-ns_k^{(n)}}\right)^{(a-\varepsilon)/s_k^{(n)}} \;\Big|\; a' < s_k^{(n)} < a'+\varepsilon'\right].$$

Using that $a' < s_k^{(n)}(\mathcal{M}) < a'+\varepsilon'$ the asymptotical behavior of $p_n(k,a')$ is then

$$\limsup_{n\to\infty}\frac{1}{n}\log_2 p_n(k,a') = \limsup_{n\to\infty} s_k^{(n)}\frac{1}{ns_k^{(n)}}\log_2 p_n(k,a') = a'\cdot\left(F_V(a/a')-1\right) + o(\varepsilon,\varepsilon'). \tag{8.34}$$

Here, $o(\varepsilon,\varepsilon')$ is an error term which goes to zero as both, ε and ε' go to zero. If $F_V = \underline{F}_V$ the $\liminf_n$ equals the same expression (8.34). We can not conclude, however, that the limit exists because $p_n(k,a')$ depends also on ε and ε'. Nevertheless, we have then sufficient control on the asymptotical behavior of $p_n(k,a')$ for all n.

Let us stop for a moment and redo this calculation in the special case of V being an H-sssi process. We may proceed then similar as in (7.11) with the only difference that V is now considered over an interval of length $2^{-ns_k^{(n)}(\mathcal{M})}$ in the case of $s_k^{(n)}(\mathcal{V}) = \alpha_k^{(n)}(\mathcal{V})$ (similarly for $h_k^{(n)}(\mathcal{V})$). Using stationarity of increments and self-similarity of V leads to

$$\begin{aligned} p_n(k,a') &= P_\Omega[2^{-n(a+\varepsilon)} < |V(2^{-ns_k^{(n)}(\mathcal{M})})| < 2^{-n(a-\varepsilon)} \;\big|\; s_k^{(n)}(\mathcal{M})] \\ &= P_\Omega[n(a-H\cdot s_k^{(n)}(\mathcal{M})-\varepsilon) < s_0^{(0)}(\mathcal{V}) < n(a-H\cdot s_k^{(n)}(\mathcal{M})+\varepsilon) \;\big|\; s_k^{(n)}(\mathcal{M})]. \end{aligned}$$

Using the approximation $s_k^{(n)}(\mathcal{M}) \approx a'$ and proceeding as in (7.12) gives

$$\limsup_{n\to\infty}\frac{1}{n}\log_2 p_n(k,a') = \begin{cases} (a-H\cdot a')\underline{q} + o(\varepsilon,\varepsilon') & \text{if } H\cdot a'+\varepsilon < a, \\ o(\varepsilon,\varepsilon') & \text{if } a = H\cdot a', \\ (a-H\cdot a')\overline{q} + o(\varepsilon,\varepsilon') & \text{if } H\cdot a'-\varepsilon > a. \end{cases}$$

A moment's thought reveals that this is indeed $a'\cdot(F_V(a/a')-1)+o(\varepsilon,\varepsilon')$ as before.

Let us now continue the proof. Since the asymptotics of $p_n(k,a')$ does not depend on k we may write $p_n(l\varepsilon') = p_n(k,l\varepsilon')$ and group terms as follows

$$\pi_n := \sum_{k=0}^{2^n-1} P_\Omega[a-\varepsilon < s_k^{(n)}(\mathcal{V}) < a+\varepsilon \;\big|\; \mathcal{M}] = \sum_l p_n(l\varepsilon')\cdot N_n^{\mathcal{M}}(l\varepsilon',\varepsilon'/2). \tag{8.35}$$

This approach closely follows the techniques of the proof of theorem 4.2 and lemma 3.6. First, one notices that for fixed ε and ε' there are only finitely many indices l to consider. Hence the asymptotic behavior of (8.35) is dominated by the asymptotically largest term appearing in the sum.

The asymptotics of p_n (8.34) combined with the definition of $f_{\mathcal{M}}$ (1.8) yield

$$\limsup_{n\to\infty}\frac{1}{n}\log_2\left[p_n(a')\cdot N_n^{\mathcal{M}}(a',\varepsilon'/2)\right] = f_{\mathcal{M}}(a') + a'\cdot(F_V(a/a')-1) + o(\varepsilon,\varepsilon'). \tag{8.36}$$

As in (3.16) and (4.4) one concludes that

$$\limsup_{n\to\infty} \frac{1}{n}\log_2 \pi_n = \sup_{a'} \left(f_{\mathcal{M}}(a') + a' \cdot (F_V(a/a') - 1)\right) + o(\varepsilon, \varepsilon').$$

For this computation one uses the general fact that $\liminf(x_n) + \liminf(y_n) \le \liminf(x_n + y_n) \le \liminf(x_n) + \limsup(y_n) \le \limsup(x_n + y_n) \le \limsup(x_n) + \limsup(y_n)$. Because some of these inequalities may be strict, however, one needs to know that at least one of $f_{\mathcal{M}} = \underline{f}_{\mathcal{M}}$ or $F_V = \underline{F}_V$ holds to obtain this limsup; and both equalities have to hold if one likes to estimate the liminf. For H-sssi processes, $F_V = \underline{F}_V$ follows from corollary 7.5. Letting ε and ε' tend to 0 we get, finally,

$$F_{\mathcal{V}}(a) \bigm|_{\mathcal{M}} = \sup_{a'} \left(f_{\mathcal{M}}(a') + a' \cdot (F_V(a/a') - 1)\right) \tag{8.37}$$

Formula (8.37) captures the *multifractal interaction* between warp time and warped motion. It makes most explicit how the combination of singularity exponents of $\mathcal{M}$ and V results in the spectrum of $\mathcal{V} = V(\mathcal{M})$. The RHS of (8.37) is the announced definition of $f^{\triangle}_{\mathcal{M}}$ in all generality (compare (8.24)). We will discuss its properties as well as the fact why we consider it a transform of $f_{\mathcal{M}}$ rather than of F_V momentarily.

The proof of the first part of theorem 8.12 is complete now. For the second part, we write by the law of total probability, by (3.22) and by rearranging terms,

$$\begin{aligned} \sum_{k=0}^{2^n-1} P_{\Omega}[\,|s_k^{(n)}(\mathcal{V}) - a| < \varepsilon] &= \sum_{k=0}^{2^n-1} \sum_{l} p_n(l\varepsilon') \cdot P_{\Omega}[\,|s_k^{(n)}(\mathcal{M}) - l\varepsilon'| < \varepsilon'/2] \\ &= \sum_{l} p_n(l\varepsilon') \cdot \mathbb{E}_{\Omega}[N_n^{\mathcal{M}}(l\varepsilon', \varepsilon'/2)] \end{aligned}$$

Comparing with (8.35) and recalling (3.23) it is clear that the same analysis as above applies, with $f_{\mathcal{M}}$ replaced by $F_{\mathcal{M}}$. This proofs (8.28). ◇

Proof of corollary 8.13

To proof the corollary we apply theorem 3.14 conditioned on knowing $\mathcal{M}$. Note that the singularity exponents $s_k^{(n)} = |L_H(\mathcal{M}((k+1)2^{-n})) - L_H(\mathcal{M}(k2^{-n}))|$ $(k = 1, 2, \ldots)$ become independent when conditioned on $\mathcal{M}$. Similarly for $s_k^{(n)} = \sup\{|L_H(u) - L_H(u') : \mathcal{M}((k-1)2^{-n}) < u < u' < \mathcal{M}((k+2)2^{-n})|$, at least in the sense of remark 3.15. Finally, (8.30) follows from the multifractal formalism corollary 4.1 applied to $\mathcal{V}$ and from $f_{\mathcal{V}} = f^{\triangle}_{\mathcal{M}} = T^{\|}_{\mathcal{M}} = T_{\mathcal{V}}$. ◇

General definition of $\boldsymbol{f}^{\triangle}$

Motivated by formula (8.37) we extend definition 8.10:

Definition 8.14. *The* Y-warped spectra of $\mathcal{M}$ *are given by*

$$f^{\triangle}_{\mathcal{M}}(a) = \sup_{a'} \left(f_{\mathcal{M}}(a') + a' \cdot (F_Y(a/a') - 1)\right) \tag{8.38}$$

and similar for $F_{\mathcal{M}}^{\triangle}$.

Let $\hat{a}$ mark a value for which F_Y attains its maximal value, which is always 1. Then, $(F_Y(a/a')-1)=0$ at $a'=a/\hat{a}$ which implies that

$$f_{\mathcal{M}}^{\triangle}(a) \geq f_{\mathcal{M}}(\frac{a}{\hat{a}}).$$

In order to get a more graspable idea of $f^{\triangle}$ consider the two-parameter family of functions $g_{a,z}(a') := z - a' \cdot (F_Y(a/a')-1)$. First, observe that each such function has as minimum point in $(a/\hat{a}, z)$. Second, since $f_{\mathcal{M}}(a') \leq 1$ we have $g_{a,z} > f_{\mathcal{M}}$ for $z > 1$.

We are now ready to give an explanation of (8.38) as a 'touching' or 'morphological' transform (compare Figure 9):

$$f_{\mathcal{M}}^{\triangle}(a) = \min\{z \,:\, g_{a,z} > f_{\mathcal{M}}\}. \tag{8.39}$$

In other words, $z = f_{\mathcal{M}}^{\triangle}(a)$ is the critical value of z for which the curves $g_{a,z}$ and $f_{\mathcal{M}}$ touch.

Example 8.3. (Warped fBm) In the special case of $Y = B_H$ and choosing the Hölder exponents $h_k^{(n)}$, the spectrum F_Y reduces to the point $(\hat{a}, 1)$ and we find as earlier

$$f_{\mathcal{M}}^{\triangle}(a) = f_{\mathcal{M}}(a/\hat{a}) \qquad (\text{if } \underline{q} = -\infty, \overline{q} = \infty.) \tag{8.40}$$

♠

Example 8.4. (Warped H-sssi process) For an H-sssi process Y and general $s_k^{(n)}$ one has $\hat{a} = H$ and by (7.13)

$$g_{a,z}(a') = \begin{cases} z - (a - H \cdot a')\underline{q} & \text{for } a' \leq a, \\ z - (a - H \cdot a')\overline{q} & \text{for } a' > a. \end{cases} \tag{8.41}$$

Note that $\underline{q} < 0 < \overline{q}$. Thus, $g_{a,z}$ has indeed the shape of Figure 9 (a), with a unique minimum in $(a/H, z)$ and with slopes $H\underline{q}$ and $H\overline{q}$ to the left and right of this minimum, respectively.

As we have seen in (8.39) one needs to draw the 'touching' $g_{a,z}$ and to follow their minima. Keeping the simple form of $g_{a,z}$ in mind we may conclude as follows:

If a is such that the appropriate $g_{a,z}$ is touching $f_{\mathcal{M}}$ with its minimum, then we have $f_{\mathcal{M}}^{\triangle}(a) = f_{\mathcal{M}}(a/H)$.

For all other a the appropriate $g_{a,z}$ is touching $f_{\mathcal{M}}$ with one of its linear parts, the small (large) a on the one of slope $H\overline{q}$ $(H\underline{q})$. Using (8.39) and $z = g_{a,z}(a/H)$ it follows that $f_{\mathcal{M}}^{\triangle}(a) = z$ depends linearly on a, for small (large) a with slope $\overline{q}$ $(\underline{q})$.

Intuitive recipe Since an H-warped $f_{\mathcal{M}}^{\triangle}$ is continuous, computing it amounts to finding 'tangents' to $f_{\mathcal{M}}$ of slope $H\overline{q}$ and $H\underline{q}$ on the left and right, respectively. Following the graph of $f_{\mathcal{M}}$ in its center part and the tangents at the outskirts on gets a plot of $f_{\mathcal{M}}^{\triangle}(H\cdot)$.

Rigorous recipe One takes the Legendre transform of $f_{\mathcal{M}}(\cdot/H)$ (which is nothing but $\tau_{\mathcal{M}}(\cdot H)$), cuts off the slopes q outside the interval $(\underline{q}, \overline{q})$ and transforms back. Thus, (8.38) is a true extension of (8.24). ♠

Note that $f^{\triangle}$ is not always concave, though it may remove some of the non-concave parts of f. For toy models of non-concave f which will lead to non-concave $f^{\triangle}$ see [91, 94].

Finally, let us explain why (8.38) should be viewed as a transform of $f_{\mathcal{M}}$ rather than one of F_Y. First, the simplicity of (8.41) and (8.24) may be misleading. In general, the smoothing 'kernel' $a' \cdot (F_Y(a/a') - 1)$ will change its shape as a changes. A second reason is found in (8.40). Third, F_Y is deterministic and $f_{\mathcal{M}}$ is not.

Example 8.2. (Lévy subordinator, revisited) Let us apply formula (8.38) to the situation introduced in Example 8.2 on page 76 and compute the *positive* part of $f^{\triangle}_{L_{H'}}$. Recall that $\overline{q} = 1/H$ and $\underline{q} = -1$ for the singularity exponent $\alpha_k^{(n)}$ considered there. According to (7.17) the graph of $f_{L_{H'}}(\cdot/H)$ consists of two line segments joining $(HH', 1)$ with the origin and with $(H(H'+1), 0)$ (slopes $1/(HH')$ and $-1/H$).

At this point it is already clear that $f_{\mathcal{V}}(HH') = f^{\triangle}_{L_{H'}}(HH') = f_{L_{H'}}(H') = 1$ which marks the peak of $f_{\mathcal{V}}$. This does not come as a surprise: the (Lebesgue) almost everywhere present Hölder exponents H and H' of motion and time change (see [54, Rem. 4]) combine into the (Lebesgue) almost everywhere present Hölder exponent HH' of the compound process.

Let us assume $H' > 1$ in order to have an increasing warp time. Then, $f_{\mathcal{V}} = f^{\triangle}_{L_{H'}} = f_{L_{H'}}(\cdot/H)$ in their left part, i.e., for $a < HH'$. This is because their slope is there equal to $1/(HH')$ which lies in the interval $(\underline{q}, \overline{q}) = (-1, 1/H)$. Consequently, $f_{\mathcal{V}}$ will pass through the origin.

If $H > 1$, then $f_{\mathcal{V}} = f^{\triangle}_{L_{H'}} = f_{L_{H'}}(\cdot/H)$ also in the right part since the slope is there $-1/H$ which is within $(\underline{q}, \overline{q}) = (-1, 1/H)$:

$$\mathcal{V} = L_{H>1}(L_{H'>1}) : \qquad f_{\mathcal{V}}(a) = \begin{cases} 1 + (a - H \cdot H')/(H \cdot H') & \text{if } 0 \le a \le H \cdot H' \\ 1 - (a - H \cdot H')/H & \text{if } a > H \cdot H'. \end{cases}$$

However, if $H < 1$ then the right part of $f^{\triangle}_{L_{H'}}$ will have slope -1, the smallest one allowed:

$$\mathcal{V} = L_{H<1}(L_{H'>1}) : \qquad f_{\mathcal{V}}(a) = \begin{cases} 1 + (a - H \cdot H')/(H \cdot H') & \text{if } 0 \le a \le H \cdot H' \\ 1 - (a - H \cdot H') & \text{if } a > H \cdot H'. \end{cases}$$

With (8.38) we have now a better understanding of the difference of these two cases. First, when $H < 1$ the subordinated process $L_H(L_{H'})$ is somewhat *more regular* than a comparison with fBm would make one think, i.e., $f^{\triangle}_{L_{H'}} > f_{L_{H'}}(\cdot/H)$ for $a > HH'$ (note that this agrees with (8.25)). For an explanation note that the 'touching point' of $f_{L_{H'}}$ with $a'(F_{L_H}(a/a') - 1)$ occurs at the peak of $f_{L_{H'}}$ meaning that the (Lebesgue) almost everywhere present $L_{H'}$-points with $h(t) = H' > 1$ control the entire right side of the multifractal spectrum of $\mathcal{V}$. The left part is governed by the (Lebesgue) almost everywhere present L_H-points with $h(t) = H$. Second, in the case $H > 1$ the entire spectrum is controlled by the same (Lebesgue) almost everywhere present L_H-points with $h(t) = H$. ♠

8.5 Multifractals in Multifractal Time

In Section 8.4 we actually established theorem 8.12 not only for H-sssi processes V but for general processes Y with stationary increments and some regular multifractal scaling:

Theorem 8.15 (Multifractals in multifractal time). *Let Y and $\mathcal{M}$ be independent processes and set $\mathcal{Y}(t) = Y(\mathcal{M}(t))$. Assume that the distributions of $Y(t+\delta) - Y(t)$ are independent of t. Admit the singularity exponents $\alpha_k^{(n)}$, $h_k^{(n)}$ and $w_k^{(n)}$ under the same conditions as in theorem 8.5. Assume that $F_Y = \underline{F}_Y$. Then, we have the* **warp formula**

$$F_{\mathcal{Y}}(a)\,\big|_{\mathcal{M}} = f_{\mathcal{M}}^{\triangle}(a) \qquad \textit{and} \qquad F_{\mathcal{Y}}(a) = F_{\mathcal{M}}^{\triangle}(a), \tag{8.42}$$

and[††]

$$T_{\mathcal{Y}}(q)\,\big|_{\mathcal{M}} = \tau_{\mathcal{M}}(T_Y(q)+1) \qquad\qquad T_{\mathcal{Y}}(q) = T_{\mathcal{M}}(T_Y(q)+1). \tag{8.43}$$

Moreover, if $f_{\mathcal{M}} = \underline{f}_{\mathcal{M}}$ then $F_{\mathcal{Y}}\big|_{\mathcal{M}} = \underline{F}_{\mathcal{Y}}\big|_{\mathcal{M}}$.

Proof
To obtain $T_{\mathcal{Y}}$ we simply take the Legendre transform in (8.42) . Assuming $a' > 0$ we find

$$\begin{aligned}
T_{\mathcal{Y}}(q)\Big|\mathcal{M} = F_{\mathcal{Y}}^{*}(q)\Big|\mathcal{M} &= \inf_a \Big(qa - F_{\mathcal{Y}}(a)\Big|\mathcal{M}\Big) = \inf_a \Big(qa - f_{\mathcal{M}}^{\triangle}(a)\Big) \\
&= \inf_a \Big(qa + \inf_{a'} \left(-f_{\mathcal{M}}(a') - a'(F_Y(a/a') - 1)\right)\Big) \\
&= \inf_{a'} \Big(-f_{\mathcal{M}}(a') + a'\Big(1 + \inf_a \Big(q\frac{a}{a'} - F_Y(\frac{a}{a'})\Big)\Big)\Big) \\
&= \inf_{a'} \left(-f_{\mathcal{M}}(a') + a'\left(1 + T_Y(q)\right)\right) \\
&= f_{\mathcal{M}}^{*}(1 + T_Y(q)) = \tau_{\mathcal{M}}(1 + T_Y(q)).
\end{aligned}$$

With similar manipulations one obtains $T_{\mathcal{Y}}$. As a matter of fact, (8.43) can easily be obtained directly using the technique of (8.33). ♢

Formula (8.43) shows how to extend definition 8.9:

Definition 8.16. *The* Y-warped partition functions of $\mathcal{M}$ *are given by*

$$T_{\mathcal{M}}^{\|}(q) := T_{\mathcal{M}}(T_Y(q)+1) \tag{8.44}$$

and similarly for $\tau_{\mathcal{M}}^{\|}(q)$.

[††]Formula (8.43) reads the same for any c-ary —not only dyadic— scaling (compare footnote ‡). If we replaced, however, $S^{(n)}(q)$ by $\mathbb{E}_n\left[2^{-nqs_k^{(n)}}\right] = c^{-dn}S^{(n)}(q)$ in the definition of $T(q)$ (compare (3.11) and (3.19)) then (8.43) would become the appealing

$$T_{Y(\mathcal{M})} = T_{\mathcal{M}}(T_Y).$$

Indeed, since for H-sssi processes V we have $T_V(q) = qH - 1$ for $\underline{q} < q < \overline{q}$ and may indeed write (8.22) in the form $T^{\|}_{\mathcal{M}}(q) = T_{\mathcal{M}}(T_V(q) + 1)$ in that case.

Example 8.5. (Subordinated Multifractals) Turning around the usual rôles we may consider $\mathcal{Y} = \mathcal{M}(L_H)$ and get

$$T_{\mathcal{M}(L_H)}(q) = \begin{cases} H \cdot T_{\mathcal{M}}(q) + H - 1 & \text{if } \underline{q} < T_{\mathcal{M}}(q) + 1 < \overline{q} \\ -\infty & \text{else.} \end{cases}$$

Assuming differentiability this leads with (4.11) to $F_{\mathcal{M}(L_H)}(a) = HF_{\mathcal{M}}(a/H) + 1 - H$ which is valid for $a = HT'_{\mathcal{M}}(q)$ with q in the above range; for other a, $F_{\mathcal{M}(L_H)}(a)$ will be linear of slope such as to make the graph smooth for all a. ♠

Combining theorem 8.15 and 3.14 we get:

Corollary 8.17 (Lévy processes in multifractal time). *Let Y and $\mathcal{M}$ be as in theorem 8.15 with the additional assumption that the increments of Y are independent. Then, we have the pathwise* **warp formula**

$$f_{\mathcal{Y}}(a, \omega) \Big|_{\mathcal{M}} \overset{\text{a.s.}}{=} F_{\mathcal{Y}}(a) \Big|_{\mathcal{M}} = f^{\triangle}_{\mathcal{M}}(a). \tag{8.45}$$

Moreover, the **multifractal formalism**

$$f_{\mathcal{Y}}(a, \omega) \overset{\text{a.s.}}{=} \tau^*_{\mathcal{Y}}(a, \omega) \overset{\text{a.s.}}{=} T^*_{\mathcal{Y}}(a) \tag{8.46}$$

holds for any a for which $\left(T^{\|}_{\mathcal{M}}\right)^*(a) \overset{\text{a.s.}}{=} \underline{f}^{\triangle}_{\mathcal{M}}(a) > 0.$

A process Y with stationary and independent increments as in corollary 8.17 is called a *Lévy process*. In the special case when Y is γ-stable, it is necessarily $1/\gamma$-sssi, in other words Lévy stable motion, which we dealt with earlier. Jaffard in [54] has obtained the almost sure dimension based spectrum for a large class of Lévy processes, in particular Lévy stable motion. Here, we computed the grain based spectra of warped Lévy processes, provided one has knowledge on the moments or the tail behavior of the marginals of Y in order to compute T_Y or F_Y.

Example 8.6. (Lévy motion in multifractal time) A special case, in which all assumptions of corollary 8.17 hold, is present if the warped process is Lévy stable motion, i.e., $Y = L_H$, and where the warp time is a binomial cascade, i.e., $\mathcal{M} = \mathcal{M}_b$. This is true for the singularity exponents $h^{(n)}_k$ and $\alpha^{(n)}_k$ without further ado, while for $w^{(n)}_k$ one has to check the existence of moments of negative order for $A_{n,k}$ (compare corollary 6.2). ♠

Dimension based spectra

There is a simple class of multifractals in multifractal time on $[0, 1]$ for which the full multifractal formalism, in particular the dimension based spectra are known. As we will see the same warp formula (8.43) will hold despite the fact that strictly speaking the example does not fit in the above framework.

Full multifractal warp formalism Let $\mathcal{M}_b$ and $\mathcal{K}_b$ be *independent* binomial distribution functions as in (5.2) with properties (i)-(iii) (see (5.5)) and with multipliers $M_k^{(n)}$ and $K_k^{(n)}$, respectively. To ensure *stationarity* at least of their dyadic increments assume in addition that $M_0 \stackrel{\mathrm{d}}{=} M_1$ and $K_0 \stackrel{\mathrm{d}}{=} K_1$.

Now, let $\mathcal{K}_b^\dagger$ be the *inverse* function of $\mathcal{K}_b$ just as in Section 5.6, i.e., $\mathcal{K}_b(\mathcal{K}_b^\dagger(t)) = t$. Let $\mathcal{Y} = \mathcal{M}_b(\mathcal{K}_b^\dagger)$. Since $\mathcal{M}_b$ and $\mathcal{K}_b^\dagger$ are both continuous and strictly increasing, so is $\mathcal{Y}$ and we may define a measure by setting $\mu([0,t]) := \mathcal{Y}(t)$. Let us check at how μ distributes mass on the unit interval $[0,1]$. Therefore, let $t_{n,k} := \mathcal{K}_b(k2^{-n})$. These points partition the unit interval into 2^n intervals $J_k^{(n)} := [t_{n,k}, t_{n,k+1}]$ which have the length $|J_k^{(n)}| = t_{n,k+1} - t_{n,k}$ equal to

$$\mathcal{K}_b((k_n+1)2^{-n}) - \mathcal{K}_b(k_n 2^{-n}) = K_{k_n}^{(n)} \cdot K_{k_{n-1}}^{(n-1)} \cdots K_{k_1}^{(1)} \cdot K_0^{(0)}$$

and μ-mass $\mu(J_k^{(n)}) = Y(t_{n,k+1}) - Y(t_{n,k})$ equal to

$$\mathcal{M}_b((k_n+1)2^{-n}) - \mathcal{M}_b(k_n 2^{-n}) = M_{k_n}^{(n)} \cdot M_{k_{n-1}}^{(n-1)} \cdots M_{k_1}^{(1)} \cdot M_0^{(0)}.$$

Thus, we recognize in this measure μ immediately the measure μ_c of Section 5.6 with $c = 2$ and $L_i^{(n)} = K_i^{(n)}$. In other words, μ is a special case of a so-called statistically self-similar measure [33, 6, 8] for which equality in the multifractal formalism for the singularity exponents $\alpha_k^{(n)}$ and $h_k^{(n)}$ is well established, and for which T is given through the implicit equation (compare (5.41))

$$\mathbb{E}\left[M_0^q K_0^{-T(q)} + M_1^q K_1^{-T(q)}\right] = 2\mathbb{E}\left[M^q K^{-T(q)}\right] = 2\mathbb{E}\left[M^q\right]\mathbb{E}\left[K^{-T(q)}\right] = 1. \quad (8.47)$$

For a comparison of (8.47) with the usual warp formula (8.43) let us recall that

$$\mathbb{E}[M^q] = 2^{-T_{\mathcal{M}_b}(q)-1} \qquad \text{and} \qquad \mathbb{E}[K^{-T_{\mathcal{K}_b^\dagger}(q')}] = 2^{q'-1} \quad (8.48)$$

from (5.25) and (5.40). Setting $q' = T_{\mathcal{M}_b}(q)+1$ in the last expression we see that (8.47) is indeed solved by $T(q) = T_{\mathcal{K}_b^\dagger}(T_{\mathcal{M}_b}(q)+1)$. We see a reason for (8.43) to hold in the fact that, though $\mathcal{M}_b$ does not have stationary increments of all lengths, nevertheless the lengths $|J_k^{(n)}|$ and the masses $\mu(J_k^{(n)})$ are stationary, i.e., of equal distribution for $k = 0, \ldots, 2^n - 1$ (n fixed).

Remark 8.18. (Independence of warp time and warped motion) Note that (8.43) does not necessarily hold if Y and $\mathcal{M}$ are dependent. This follows, e.g., when choosing $\mathcal{K}_b$ to be pathwise equal to $\mathcal{M}_b$. Then, we get $\mathcal{Y}(t) = t$ and $T_{\mathcal{Y}}(q) = q - 1$. Combining $T_{\mathcal{K}_b}(-T_{\mathcal{K}_b^\dagger}(q)) = -q$ (which follows from (8.48)) with (8.43) we find $T_{\mathcal{K}_b}(1-q) = T_{\mathcal{K}_b}(-T_{\mathcal{Y}}(q)) = -T_{\mathcal{M}_b}(q) - 1 = -T_{\mathcal{K}_b}(q) - 1$. Setting $q = 1/2$ gives $T_{\mathcal{K}_b}(1/2) = -1/2$. Since $T_{\mathcal{K}_b}(0) = -1$ and $T_{\mathcal{K}_b}(1) = 0$ we conclude that $T_{\mathcal{K}_b}$ must be linear for $0 \le q \le 1$ due to concavity. But this does not hold in general and we must conclude that (8.43) may fail when the components are dependent. ♣

Warped, inverse and relative multifractals

In the framework of so-called *relative multifractal analysis* one aims at obtaining more flexible multifractal spectra by replacing the usual singularity exponents of a process $\mathcal{M}$, say

$$\alpha_k^{(n)}(\mathcal{M}) = \frac{\log|\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})|}{\log|I_k^{(n)}|} = \frac{\log|\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})|}{-n\log(2)},$$

by exponents which take into account a different, hopefully more sensitive measure of $I_k^{(n)}$ than its length. More explicitly, the *multifractal spectra of $\mathcal{M}_b$ relative to $\mathcal{K}_b$* are in terms of the singularity exponents

$$r_k^{(n)} = \frac{\log|\mathcal{M}_b((k+1)2^{-n}) - \mathcal{M}_b(k2^{-n})|}{\log|\mathcal{K}_b((k+1)2^{-n}) - \mathcal{K}_b(k2^{-n})|}.$$

In the notation of [95] $r^{\mathcal{M}_b/\mathcal{K}_b}(\theta) = \lim_n r_{k_n}^{(n)}$ $(k_n 2^{-n} \to \theta)$. Comparing with $\mathcal{Y} = \mathcal{M}_b(\mathcal{K}_b^\dagger)$ from above we find immediately that $\alpha_k^{(n)}(\mathcal{Y}) = \log\mu(J_k^{(n)})/\log|J_k^{(n)}|$ coincides with the 'relative exponent' $r_k^{(n)}(\mathcal{M}_b/\mathcal{K}_b)$. The only difference is that they 'live' in two different spaces: $r_k^{(n)}$ is defined for θ-intervals $I_k^{(n)}$, and $\alpha_k^{(n)}$ for t-intervals $J_k^{(n)} = \mathcal{K}_b(I_k^{(n)})$.

Nevertheless, we may immediately say that the number of intervals $I_k^{(n)}$ of length 2^{-n} with $r_k^{(n)} \simeq a$ grows as $(2^{-n})^{-f(a)}$ (compare (3.5)) where $f = f_{\mathcal{M}_b(\mathcal{K}_b^\dagger)} = T^*$ from (8.47). This is exactly the formula '$f_R(\alpha) = \beta^*(\alpha)$' of [95, p. 160] and can also be found in [63].

However, it might be appropriate to abandon the Euclidean length $|I_k^{(n)}|$ in θ-space altogether and to replace it by $|\mathcal{K}_b((k+1)2^{-n}) - \mathcal{K}_b(k2^{-n})|$ not only what concerns the singularity exponent but also in the spectra. Again, (8.37) is key. It helps understand how to do so as it shows how to transfer resolution from one space to the other: the value of a' for which the supremum is attained is the predominant distortion factor by which $\mathcal{K}_b^\dagger$ 'expands' dimension. This suggests the formula $f_{r_k^{(n)},\mathcal{M}_b}(a) = a' \cdot f_{\mathcal{M}_b(\mathcal{K}_b^\dagger)}(a)$. It holds indeed in the form '$\dim(K_\alpha^{\mathcal{M}_b/\mathcal{K}_b}) = \beta^*(\alpha)\cdot\alpha^{\mathcal{K}_b}(q)$' of [95, p. 160].

Choosing $\mathcal{K}_b$ to be iid with $\mathcal{M}_b$ this formula suggests what to expect of a multifractal analysis of a random measure relative to an independent realization of itself.

Understanding the multifractal structure of $\mathcal{M}_b$ with respect to $\mathcal{K}_b$ is, thus, essentially equivalent to understanding the structure of $\mathcal{M}_b$ warped by $\mathcal{K}_b^\dagger$, the *inverse* multifractal of $\mathcal{K}_b$. Since inverse measures can be defined in general [94] and since the basis of the relative multifractal formalism has been expanded [17] this principle is bound to hold in far broader context.

Conclusion

We have presented the coarse grain aspects of multifractal analysis from a most general point of view which allows to connect the various different approaches currently employed in the literature, i.e., various singularity exponents as well as different notions of dimensions and scaling.

The framework presented proves convenient for exploring the multifractal properties of the well known random cascade measures as well as self-similar processes. But it applies also to the novel compound processes we introduced. In subordinating fractional Brownian motion (fBm) – and more generally stable motion – to multifractal time we have only scratched the surface of a new class of multifractal objects. Many questions must jump to mind immediately which are beyond the scope of this paper.

A problem of general interest is the construction of multifractal *measures* with stationary increments. In [14, 70, 74, 105] different classes of such processes, each with appealing properties, are proposed.

From a multifractal point of view one can ask for more precise spectral information such as exact Hausdorff gauge functions and Hausdorff measures of sample paths, an issue which has been solved for graphs and images of fBm only in the last years [107, 108].

In view of the dimension based multifractal spectra it would be interesting to know to what extent equality in the multifractal formalism holds. A first approach to this problem could be to focus on time change of Lévy processes.

From a simulation or engineering point of view the new models raise the question of efficient and accurate estimation of parameters and synthesis. There is a general need for reliable empirical evidence of multifractal properties of time series, as well as for plausible, physical explanations of multiplicative structures towards better understanding. Research in this area could profit from the large literature discussing the extraction of multifractal parameters out of physical systems which took its beginning with [42, 45, 46]. On the statistics of multifractal estimators from time series some work has been done recently [41, 3, 40].

The modelling of fractal and multifractal processes has so far relied mostly on fBm and cascades. For TCP traffic loads and web access file size processes, e.g., multifractal cascades such as the binomial provide accurate, yet parsimonious models and have had a considerable impact [89, 96, 38, 88]. Other traffic loads which are not subject to large time delay, such as video, are apparently well modelled by fBm. Finally, for financial data, fBm in multifractal time has been proposed as a model in [68].

For simulation purposes of fBm in multifractal time a good idea is to start with Wiener motion which has independent increments. Alternatively, one may submit the randomized Weierstrass function to multifractal time since it is more simple to synthesize than fBm. Moreover, modifying approximate models based in the wavelet domain such as WIG (see [88]) or the FFT based models of [20] should be more practical than direct synthesis in the time domain in the presence of the strong correlations of fBm.

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