On non scale invariant Infinitely Divisible Cascades

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Résumé – Nous présentons les définitions et synthèses de processus stochastiques respectant des lois d'échelles qui s'écartent de façon contrôlée d'un comportement en loi de puissance. Nous définissons des bruit, mouvement et marche aléatoire issus de cascades infiniment divisibles (IDC). Nous étudions analytiquement le comportement des moments des accroissements de ces processus à travers les échelles. Ces résultats théoriques sont illustrés sur l'exemple d'une cascade log-Normale non invariante d'échelle. Les algorithmes de synthèse et les fonctions MATLAB utilisés sont disponibles sur nos pages web.

Abstract – We address the definition and synthesis of stochastic processes which posses scaling laws that depart from power law behaviors in a controlled manner. We define non scale invariant infinitely divisible cascading (IDC) noise, motion and random walk. We provide a theoretical derivation of the scaling behavior of the moments of their increments. The example of a non scale invariant log-Normal cascade illustrates these results. Algorithms for synthesis and MATLAB functions are available from our web pages.

1 Introduction

Scaling has been observed for many years in a large number of fields including natural phenomena: turbulence in hydrodynamics, rythm of human heart in biology, spatial repartition of faults in geology and others such as computer networks and financial markets. The multifractal formalism[14] has become one of the most popular frameworks to analyse signals that exhibit *scale invariance*. In current verbage, this term refers to the power law behavior of the absolute moments of increments $\delta_\tau X(t) = X(t+\tau) - X(t)$ of a process X. Then, scaling invariance is to be described by a set of multifractal exponents $\zeta(q)$ such that[1]

$$\mathbb{E}|\delta_\tau X(t)|^q = C_q \tau^{\zeta(q)} \qquad \text{as } \tau \to 0. \tag{1}$$

For instance, statistically self-similar processes such as fractional Brownian motions [11] with Hurst exponent H fit into this framework with $\zeta(q) = qH$.

In real world applications, one usually confines to observing power laws in a given range of scales $\tau_{min} \le \tau \le \tau_{max}$ as a best approximation to (1) and to the multifractal formalism which originally aims at the description of singularities in the trajectories of processes (compare (1)). We refer to such behavior as *multiscaling* to distinguish it from multifractals. The need for an appropriate mathematical framework substituting (1) is met with the one of the *infinitely divisible cascades* (IDC) [5]. It allows for more flexible scaling and thus better fitting of data and honors the contribution of all scales in a range of interesting scales $\tau_{min} \le \tau \le \tau_{max}$ as follows:

$$\mathbb{E}|\delta_\tau X(t)|^q = C_q \exp[-\zeta(q) n(\tau)],\ \tau_{min} \le \tau \le \tau_{max}, \tag{2}$$

where $n(\tau)$ is some monotonous function. Such a behavior is analysed in terms of a *cascading mechanism* through the scales from τ_{max} to τ_{min}. In terms of scale dependence, the IDC framework generalizes (1) which is recovered by choosing $n(\tau) = -\log\tau$. The difference in spirit lies in the fact that multifractal analysis applies to any process (compare footnote 1) and is concerned with local properties in the limit of fine scales, but not finite scales. Note that both, multifractal analysis and IDC scaling can be formulated using wavelet coefficients [13, 17].

While analysis tools for multiscaling processes and infinitely divisible cascades have been widely developed, only few recent works proposed tools for synthesis of processes with prescribed and controllable IDC scaling [1, 3, 9, 15, 16]. Multiplicative cascades have always played a central role to this purpose in intimate connection with multifractals. The synthesis of IDC presented below can be seen as a generalized continuous multiplicative cascade. Following a work by Barral & Mandelbrot [3] and inspired by the densified multiplicative cascades by Schmitt & Marsan [16] and the Multifractal Random Walk by Bacry *et al.* [1], we recently discussed and studied the *Infinitely Divisible Cascading processes* [6, 8]. Similar results were obtained independently and simultaneously by Bacry & Muzy [2, 12]. We extend our previous work to the case of *non scale invariant IDC* [7]. These continuous-time processes have stationary increments and exhibit continuous scaling laws with prescribed exponents (cf. $\zeta(q)$) as well as prescribed departures from power law behaviors (cf. $n(\tau) \ne -\log\tau$).

[1]A definition which works for any process is:

$$\zeta(q) = \liminf_{\tau\to 0} \log_\tau \mathbb{E}|\delta_\tau X(t)|^q.$$

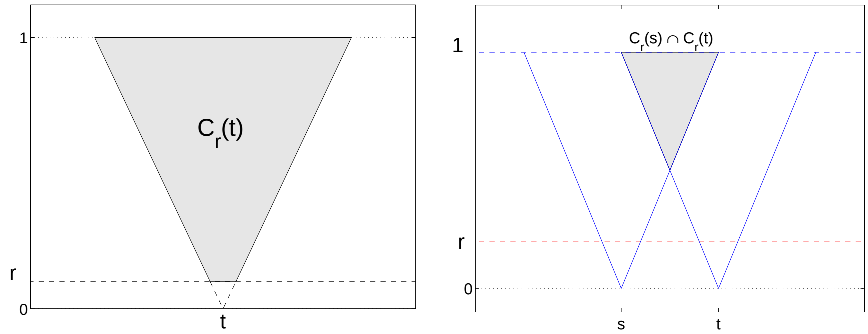


FIG. 1: **(left) Infinitely Divisible Cascade:** the geometrical "distribution of multipliers" is random, stationary in time and continuous in scale in the time-scale half-plane. **(right) Dependence between $Q_r(t)$ and $Q_r(s)$**, in particular their correlation, stems entirely from the contribution of the intersection of two cones $\mathcal{C}_r(t)$ and $\mathcal{C}_r(s)$

2 IDC Noise

Let G be an infinitely divisible distribution with moment generating function $\tilde{G}(q)$ that can be written in the form $e^{-\rho(q)}$.

Let $dm(t,r) = g(r)dtdr$ a positive measure on the time-scale half-plane $\mathcal{P}^+ := \mathbb{R} \times \mathbb{R}^+$.

Let M denote an infinitely divisible, independently scattered random measure distributed by G, and supported on the time-scale half-plane $\mathcal{P}^+$ and associated to its so-called control measure $dm(t,r)$. Random measure M is such that $\mathbb{E}[\exp[qM(\mathcal{E})]] = \exp[-\rho(q)m(\mathcal{E})]$; for all disjoints subsets $\mathcal{E}_1$ and $\mathcal{E}_2$, $M(\mathcal{E}_1)$ and $M(\mathcal{E}_2)$ are independent random variables and $M(\mathcal{E}_1 \cup \mathcal{E}_2) = M(\mathcal{E}_1) + M(\mathcal{E}_2)$.

Definition 1.
A cone of influence $\mathcal{C}_r(t)$ is defined[2] *for every $t \in \mathbb{R}$ as $\mathcal{C}_r(t) = \{(t', r') : r \le r' \le 1, t - r'/2 \le t' \le t + r'/2\}$ (see* FIG. 1*).*

An Infinitely Divisible Cascading noise *(IDC noise) is a family of processes $Q_r(t)$ parametrized by r of the form (see* FIG. 2*)*

$$Q_r(t) = \frac{\exp[M(\mathcal{C}_r(t))]}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]}. \tag{3}$$

Possible choices for distribution G are the Normal distribution, Poisson distribution, compound Poisson distributions, Gamma laws, Stable laws...

Immediate consequences of the definition are that Q_r is a stationary positive random process with:

$$\mathbb{E}Q_r = 1 \tag{4}$$

Stationarity is ensured by the specific choice of a time-invariant *control measure $dm(t,r) = g(r)dtdr$.*

Altogether, the measure M, the distribution G, the control measure m and the geometry of the cone of influence $\mathcal{C}_r(t)$ control the scaling structure as well as marginal distributions of the cascade. One major property of IDC noises is:

$$\mathbb{E}[Q_r^q] = \exp[-\varphi(q)\, m(\mathcal{C}_r)] \tag{5}$$

where

$$\varphi(q) = \rho(q) - q\rho(1), \qquad (\varphi(1) = 0), \tag{6}$$

for all q for which $\rho(q) = -\log \tilde{G}(q)$ is defined. Note the similarity between (5) and (2).

A nice property of IDC noises lies in the geometrical interpretation of their correlations that are controlled by the intersections of cones $\mathcal{C}_r(t) \cap \mathcal{C}_r(s)$ in the time-scale plane $\mathcal{P}^+$ (see FIG. 1).

3 IDC Motion & Random Walk

By analogy with T-Martingales and binomial cascades in particular, we introduce the *Infinitely Divisible Cascading Motion* as the integral of $Q_r(t)$.

Definition 2.
An Infinitely Divisible Cascading Motion *(IDC-Motion) $A(t)$ is the limiting integral*[3] *of an IDC-noise $Q_r(t)$ (see* FIG. 2*):*

$$A(t) = \lim_{r \to 0} A_r(t), \tag{7}$$

where

$$A_r(t) = \int_0^t Q_r(s)ds. \tag{8}$$

The increment process $\delta_\tau A_r(t) = A_r(t+\tau) - A_r(t)$ of A_r inherits stationarity from Q_r since $\delta_\tau A_r(t) = \int_t^{t+\tau} Q_r(s)ds$. An IDC Motion $A(t)$ inherits scaling properties from its IDC Noise $Q_r(t)$ as shown below in Section 4.

By construction, A is a non-decreasing process which appears most natural in some real world contexts, but can be seen as a severe limitation in others. Following an idea which goes back to Mandelbrot [10] and to the *Brownian motion in multifractal time*, we define a process with stationary increments, continuous scale invariance, prescribed departures from power laws and prescribed scaling exponents as well as positive and negative fluctuations: the *Infinitely Divisible Cascading Random Walk*, V_H.

Definition 3. *Let A be an infinitely divisible cascading motion, and B_H the fractional Brownian motion with Hurst parameter H. The process*

$$V_H(t) = B_H(A(t)), \quad t \in \mathbb{R}^+, \tag{9}$$

[2] Note that the large scale in the definition of $\mathcal{C}_r(t)$ has been arbitrarily set to 1 without loss of generality. Choosing a different large scale L would simply reduce to a change of units $t \to t \cdot L$, $r \to r \cdot L$.

[3] Conditions for the convergence of the positive martingale A_r as $r \to 0$ are detailed in [7].

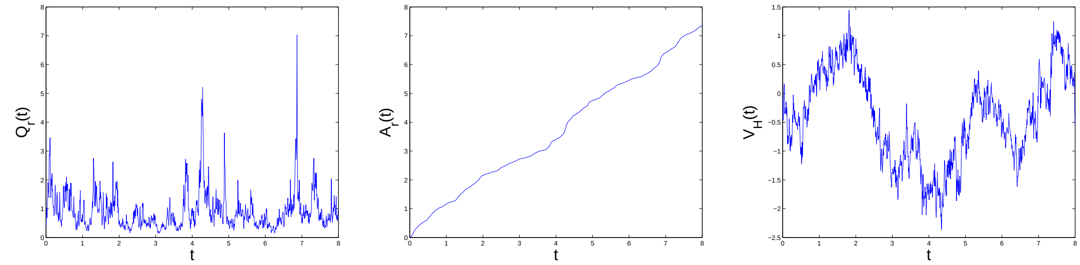


FIG. 2: Sample of a realization of **(left)** $Q_r(t)$, **(middle)** $A(t)$ and **(right)** $V_H(t)$.

is called an Infinitely Divisible Cascading Random Walk *(IDC Random Walk).*

IDC Random Walk inherits stationary increments from both B_H and A. Above all, the precise scaling behavior of $A(t)$ is transferred to $V_H(t)$ thanks to the self-similarity of the fractional Brownian motion as explained below.

4 Scaling behavior of IDC

This section states our major results: it characterizes the scaling properties of an IDC-Motion and its associated IDC-Random Walk. The reader is referred to [7] for detailed proofs. While scaling behaviors are rather easy to describe, their mathematical proof calls for some technical assumptions. Thus, we require the following notation. For an IDC Motion A_r with control measure $dm(t,r) = g(r)dtdr$ we set for $b \in (0,1)$ and $\nu > 0$

$$C_{b,\nu}[g] := \sup_{0<t\leq b} \frac{1}{t^\nu} \cdot \frac{\mathbb{E}\sup_{0\leq s\leq t}|Q_b(s)^q - Q_b(0)^q|}{\mathbb{E}[Q_b(0)^q]} \quad (10)$$

and we introduce, for $r < b^n$,

$$A_r^{(n)}(t) = \frac{1}{b^n}\int_0^{tb^n} \frac{Q_r(s)}{Q_{b^n}(s)}ds \quad (11)$$

This cascade has control measure $dm^{(n)}(t,r)$ where

$$g^{(n)}(r) := b^{2n}g(b^n r)\cdot \mathbf{1}_{[0,1]}. \quad (12)$$

Since $m^{(n)}(\mathcal{C}_{r/b^n}(s)) = m(\mathcal{C}_r(b^n s)\backslash \mathcal{C}_{b^n}(b^n s))$ we may understand $A^{(n)}$ as a zoom into the small scale details of A. In the scale invariant case ($dm(t,r) = dtdr/r^2$) we have $g^{(n)} = g$ and, thus, $A^{(n)}$ in equal in distribution to A.

Theorem
Let $q > 0$, $b \in (0,1)$ as well as the infinitely divisible law, i.e., $\rho(\cdot)$. Let A_r be an IDC Motion with control measure $g(r)dtdr$. Assume that their are constants $b \in (0,1)$ and $\nu > 0$ such that $C_{b,\nu}[g^{(n)}]$ are finite and remain bounded as $n \to \infty$; assume that A_r as well as $A_r^{(n)}$ for large n converge in $\mathcal{L}_q$. Then there exist constants $\overline{C}_q$ and $\underline{C}_q$ and $\overline{C}'_q$, $\underline{C}'_q$ such that for any $t < b$

$$\underline{C}_q t^q e^{-\varphi(q)m(\mathcal{C}_t)} \leq \mathbb{E}A(t)^q \leq \overline{C}_q t^q e^{-\varphi(q)m(\mathcal{C}_t)}, \quad (13)$$

$$\underline{C}'_q t^{qH} e^{-\varphi(qH)m(\mathcal{C}_t)} \leq \mathbb{E}|V_H(t)|^q \leq \overline{C}'_q t^{qH} e^{-\varphi(qH)m(\mathcal{C}_t)}. \quad (14)$$

The assumptions of the Theorem are verified for compound Poisson distributions as well as for the Normal distributions, assuming that the functions $g^{(n)}$ converge (see [7]).

The scaling behavior of V_H is a direct consequence of the self-similarity of a fractional Brownian motion B_H combined to the scaling behavior of an IDC Motion A. using the self-similarity of B_H, one finds that

$$\begin{aligned} \mathbb{E}[|V_H(t)|^q] &= \mathbb{E}\mathbb{E}[|B_H(A(t))|^q|A] \\ &= \mathbb{E}[|B(1)|^q]\cdot \mathbb{E}[A(t)^{qH}]. \end{aligned} \quad (15)$$

In practice, the fact that $A(t)$ and $V_H(t)$ have stationary increments and $A(0) = 0$ and $V_H(0) = 0$ yields, $\forall \tau \leq 1$,

$$\begin{cases} \mathbb{E}[\delta_\tau A^q] \sim C_q \tau^q \exp\left[-\varphi(q)m(\mathcal{C}_\tau)\right], \\ \mathbb{E}[\delta_\tau V_H^q] \sim C'_q \tau^{qH} \exp\left[-\varphi(qH)m(\mathcal{C}_\tau)\right], \end{cases} \quad (16)$$

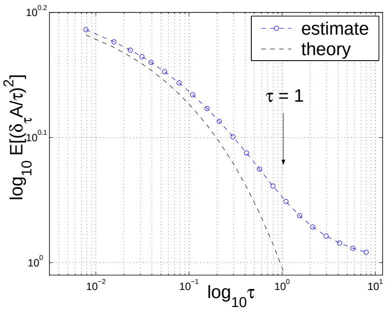
where '$\sim$' is used as a short notation for inequalities like in (13) and (14). It turns out that both sides of the '$\sim$' are close to proportional for $\tau \ll 1$. Moreover, one expects that $\mathbb{E}[\delta_\tau A^q] \sim \tau^q$ and $\mathbb{E}[\delta_\tau V_H^q] \sim \tau^{qH}$ for large $\tau \gg 1$.

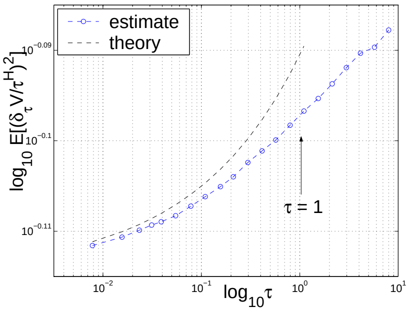
A key property of these scaling behaviors (13) or (16) is that they hold continuously through the scales, not only for a particular set of discrete scales. Again, we put the emphasis as well on the fact that the construction of Q_r and A enables a full control of the way the cascading process develops along scales and not only of the multifractal behavior obtained in the limit $\tau \to 0$. As far as applications and real world data modeling are concerned, we believe that the control of the entire cascade process is probably more relevant than that of the asymptotic behavior as $\tau \to 0$ only.

Recall that previous works[4, 5, 17] inspired a priori the search for non power law scaling of the form $\exp[-\zeta(q)n(\tau)]$ as in (2). Rather, through our approach we are naturally led in (13) and (14) to a mixture of a power law and a non power law behavior of the form $\tau^q\cdot\exp[-\varphi(q)m(\mathcal{C}_\tau)]$. This result is inherent to the use of an integral to define $A(t)$. On one hand, the $\exp\left[-\varphi(q)m(\mathcal{C}_\tau)\right]$ term is related to the underlying IDC-Noise $Q_r(t)$. On the other hand, the τ^q term is due to the fact that an IDC-Motion is obtained by *integration* of an IDC-Noise. Equation (16) does not reduce to (2) unless $n(\tau) = -\log\tau$.

5 Example

As an example of an IDC with the required properties, we propose to consider a Log-Normal cascade, i.e., distribution G is $\mathcal{N}(\mu,\sigma^2)$ and $\varphi(q) = \frac{\sigma^2}{2}q(1-q)$. We choose the control measure $dm(t,r) = 1/r^{2+\beta}dtdr$ with $\beta < 0$, which leads to the non scale invariant function $m(\mathcal{C}_\tau(0)) = (\tau^{-\beta}-1)/\beta$. This choice provably satisfies the





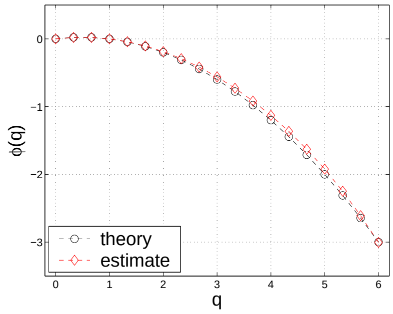


FIG. 3: **Non scale invariant cascade deviates from power laws: (left)** $\log \mathbb{E}[(\delta_\tau A/\tau)^2]$ compared to $-\varphi(2)m(\mathcal{C}_\tau)$; **(center)** $\log \mathbb{E}[(\delta_\tau V/\tau^H)^2]$ compared to $-\varphi(2H)m(\mathcal{C}_\tau)$. Scale invariance would correspond to straight lines. **(right) Prescribed exponents $\varphi(q)$ can be estimated.**

conditions of the theorem above and corresponds to the model known as the Castaing model [4] in hydrodynamic turbulence. Note that $\beta = 0$ reduces to the well-known scale invariant case ($m(\mathcal{C}_\tau) = -\log \tau$) [1, 3, 8]. Parameters of the simulation are $\mu = -0.1$, $\sigma^2 = 0.2$ and $\beta = -0.4$. The Hurst exponent H of the fractional Brownian motion B_H used to build $V_H(t)$ has been set to $H = 1/3$.

Departures from powerlaw behaviors corresponding to the $\exp[-\varphi(q)m(\mathcal{C}_\tau)]$ term in (13) are expected. FIG. 3 shows that such departures are observed on both $A(t)$ and $V_H(t)$. The performed analysis focuses on $\mathbb{E}[(\delta_\tau A/\tau)^q] \sim \exp[-\varphi(q)m(\mathcal{C}_\tau)]$, resp. $\mathbb{E}[(\delta_\tau V/\tau^H)^q] \sim \exp[-\varphi(qH)m(\mathcal{C}_\tau)]$. In a log-log plot, a curvature is clearly visible whereas the scale invariant case ($\beta = 0$) would have led to straight lines. Remark that this curvature is accurately controlled for $\tau < 1$ by the form of $m(\mathcal{C}_\tau) \neq -\log \tau$. These numerical observations are perfectly consistent with our theoretical results. Exponents $\varphi(q)$ can be estimated as well from linear regressions in $\log \mathbb{E}[(\delta_\tau A/\tau)^q]$ *vs* $m(\mathcal{C}_\tau)$ diagrams – FIG. 3(right): prescribed exponents are recovered.

The correction term to the powerlaw may be subtle, yet it is true scaling and cannot be subsumed by a constant error bound. Up to our knowledge, these are the first cascades displaying controlled non power law behaviors up to a large range of scales (two decades on FIG. 3).

In the present work, we proposed the definitions of continuous -time processes with controlled continuous multiscaling behavior. Most importantly, scaling laws exist continuously through the scales and possible departures from a power law behavior are taken into account. Up to our knowledge, Infinitely Divisible Cascading processes are the first continuous multiplicative cascades displaying controlled non power law scaling behaviors. Potential fields of application range from hydrodynamic turbulence to computer network traffic. **Matlab** routines to synthetize these processes will be available on our web pages: `www.isima.fr/~chainais`, `www-ece.rice.edu/~riedi`, `www.ens-lyon.fr/~pabry`.

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