Multifractal Formalism for Infinite Multinomial Measures

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Abstract

There are strong reasons to believe that the multifractal spectrum of DLA shows anomalies which have been termed left sided [ME, M]. In order to show that this is compatible with strictly multiplicative structures Mandelbrot et al. [M, MEH] introduced a one parameter family of multifractal measures invariant under infinitely many linear maps on the real line. Under the assumption that the usual multifractal formalism holds, the authors showed that the multifractal spectrum of these measures is indeed left sided, i.e., increasing over the whole range $\alpha_{min}, \infty]$. Here, it is shown that the multifractal formalism for self-similar measures does indeed hold also in the infinite case, in particular that the singularity exponents $\tau(q)$ satisfy the usual equation $\sum p_i \lambda_i^{\tau} = 1$ and that the spectrum $f(\alpha)$ is the Legendre transform of $\tau(q)$.

1 Introduction and Summary

In 1990, [M, MEH] introduced self-similar multifractal measures $\mu$ which are constructed with an infinitely multiplicative cascade. Data suggest that the measures considered in [MEH] show the same anomalies as observed with DLA, namely, the partition sum $\chi(q, \varepsilon)$ fails to scale like $\varepsilon^{-\tau(q)}$ for $q < 0$. This behaviour is linked with the existence of infinite Hölder exponents and with a multifractal spectrum which is increasing over the whole range $]\alpha_{min}, \infty]$, hereafter called left sided spectrum. This observation had led some authors to describe DLA as ‘non-multifractal’. On the other hand, the infinite multinomial measures (1) show that left sided spectra are compatible with strictly multiplicative, hence renormalizable structures. This paper provides a rigorous mathematical frame for the statements made in [MEH].

1Note added in proof. Infinite systems of conformal contractions are studied in [MU] comparing packing and Hausdorff measures of invariant sets.
The organization is as follows. Following Hutchinson [Hut] we will start by defining the codespace $I$, a set of sequences $i_1,i_2\ldots$ suitable to model the invariant set of a given infinite family of contractions $\{w_i\}_{i \in \mathbb{N}}$ of $\mathbb{R}^d$. There is a probability distribution on $I$ such that an address picked randomly satisfies $i_n = k$ with probability $p_k$ independent of $n$. Via the addressing this translates into a probability measure $\mu$ on $\mathbb{R}^d$ which is the only one satisfying the invariance

$$\mu = \sum_{i=1}^{\infty} p_i \mu(w_i^{-1}(\cdot)).$$

(1)

Its support is the (only) compact set satisfying

$$K = \bigcup_{i=1}^{\infty} w_i(K).$$

(2)

The necessity of taking the closure $\overline{\cdot}$ on the right hand side emphasizes the difference from the ‘classical’ case. Having established $\mu$ we proceed to estimate the Hausdorff dimension $d_{\text{HD}}(K_\alpha)$ of the sets $K_\alpha$ of points with Hölder exponent $\alpha$ (see (8) below). It is easy to show that $d_{\text{HD}}(K_\alpha)$ is bounded from above by the Legendre transform $f(\alpha)$ of the function $\tau(q)$ which is defined as follows: Let $\lambda_i := \text{Lip}(w_i)$. For given $q \in \mathbb{R}$ the implicit equation

$$\sum_{i=1}^{\infty} p_i^q \lambda_i^{-\tau} = 1$$

(3)

has either one or none solution. We define $\tau(q)$ to be the solution of (3) if there is one. Otherwise we set $\tau(q) = \infty$.

Finally, we give two conditions which together imply $d_{\text{HD}}(K_\alpha) = f(\alpha)$. First, the copies $w_i(K)$ of the support should not overlap too much. Hence, we require a separation condition similar to the well known open set condition. Secondly, we will use the spectra $f_M$ of finite approximations $\mu_M$ of $\mu$ as lower bounds of $d_{\text{HD}}(K_\alpha)$. The intuitive idea behind this is the fact that at a certain scale the details provided by the maps $w_i$ with too large $i$ can not be observed. Indeed, the convergence of $f_M$ to $f$ is in agreement with numerical measurements where one finds a sequence of preasymptotic spectra which increase to $f$ (Fig. 1). To perform the limit we require some control on the rates with which $\{p_i\}_{i \in \mathbb{N}}$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ tend to zero. We consider this second condition to be purely technical.
Figure 1: The approximation of the spectrum \( f(\alpha) \) as provided through \( f_M \) for the left sided multifractal measure of Ex. 1 for \( \kappa = .5 \), 1 and 1.5. Note that the different rate of convergence depending on the parameter \( \kappa \) is in agreement with the asymptotic behaviour of \( f(\alpha) \) for \( \alpha \to \alpha_0 \) (Fig. 2). The approximation resembles the convergence of the 'preasymptotic' spectra \( \rho_k(\alpha) + 1 \) (\( \rho_k(\alpha) \) is the probability of finding a coarse Hölder exponent \( \alpha \) at resolution \( 2^{-k} \)) as described in [M] (see also Figure 2).
2 Codespace

As pointed out with (2), having infinite many copies of a set or a measure requires one to consider the closure of sets. In other words, accumulation points have to be taken into account which can contribute considerably to the geometry of the invariant measure. Their influence, however, is captured in the geometry of the fixpoints $a_n$ of the contractions $w_n$. In order to have ‘primitive’ addresses for these fixpoints as well as for their accumulation points we identify $\mathbb{N}$ with \{a_i\}_{i\in\mathbb{N}} through $i \mapsto a_i$. Being a subset of $\mathbb{R}^d$ the set $\mathbb{N}^\ast$ of ‘points at infinity’ may contain more than one element. More precisely,

$$\mathbb{N} := \{a_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^d \quad \text{and} \quad \mathbb{N}^\ast := \mathbb{N} \setminus \{a_i\}_{i\in\mathbb{N}}. \quad (4)$$

Although $\mathbb{N}$ is embedded in space we will still think of it as being the set of natural numbers. As will be seen, it is essential for the construction of $\mu$ to assume that $\{a_i\}_{i\in\mathbb{N}}$ is bounded, i.e. that $\mathbb{N}$ is compact (see lemma 1). Equivalently, we can require that there is a bounded open set $O$, called cell, such that $w_i(O) \subset O$ for all $i \in \mathbb{N}$. We recall some separation conditions which are widely used and which apply to finite or countably infinite families \{w_i\}_{i\in J}. They are said to hold if there is a cell which satisfies the indicated condition: the open set condition (OSC)

$$w_i(O) \cap w_j(O) = \emptyset \quad \forall i \neq j \in J,$$

the disconnected open set condition (DOSC) if even the sets $w_i(\overline{O})$ ($i \in J$) are mutually disjoint and the strong open set condition (SOSC) if in addition to the OSC the cell $O$ intersects $K$.

The desired codespace is

$$I := \mathbb{N}^\ast.$$

It is convenient to define for $a \in \mathbb{N}^\ast$: $w_a(x) \equiv a$, i.e. a trivial contraction with ratio $\lambda_a := \text{Lip}(w_a) = 0$, and to set $p_a := 0$. For $i \in I$ set $\hat{i} := i_1 \ldots i_n$, $\lambda_{\hat{i}} := \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n}$, $p_{\hat{i}} := p_{i_1} \cdot \ldots \cdot p_{i_n}$, $w_{\hat{i}} := w_{i_1} \circ \ldots \circ w_{i_n}$, and, once a cell $O$ has been chosen, $V_{\hat{i}} := w_{\hat{i}}(\overline{O})$.

**Lemma 1** Assume that $\mathbb{N}$ is compact and that $\text{Lip}(w_i) = \lambda_i \to 0$ ($i \to \infty$). This is certainly satisfied if the OSC holds. Then, $\cap_{n \in \mathbb{N}} V_{\hat{i}}$ is a singleton, say $\{x_{\hat{i}}\}$, for every given $\hat{i} \in I$. Moreover, the coordinate map $\pi : I \to \mathbb{R}^d, \hat{i} \mapsto x_{\hat{i}}$ is continuous.

**Proof:** Since $V_{\hat{i}} = w_{\hat{i}_n} \circ w_{\hat{i}_{n-1}}(\overline{O}) \subset w_{\hat{i}_n}(\overline{O}) = V_{\hat{i}_{n-1}}$, the compact sets $V_{\hat{i}_n}$ form a decreasing sequence. So, their intersection cannot be empty but
has diameter zero since $|V_{j_n}| = \lambda_{j_n} \cdot |O| \to 0 \ (n \to \infty)$. This proves the first part. The continuity of $\pi$ is a direct consequence of our choice of the metric on $\mathbb{N}$, which we denote by $d(\cdot, \cdot)$. Take $x = \pi(j) \in K$ and $\varepsilon > 0$. We have to find a neighbourhood $U$ of $j$ which is mapped into $B_\varepsilon(x)$. First, there is a $n_0$ such that $V_{j_{n_0}} \subset B_\varepsilon(x)$. In case $i_k \in \mathbb{N}$ for all $n \leq n_0$ it is enough to choose $U = \{j \in I : j_k = i_k \ (1 \leq k \leq n_0)\}$. Otherwise let $n_1$ be such that $i_k \in \mathbb{N}$ for all $k < n_1$ and that $\bar{a} := i_{n_1} \in \mathbb{N}$. Since $\lambda_i \to 0 \ (i \to \infty)$, there is $m_0 \in \mathbb{N}$ s.t. $|V_k| \leq \varepsilon/3$ for all $k \geq m_0$. Now, we can choose $\delta < \varepsilon/3$ such that $m \in \mathbb{N}$ and $d(m, \bar{a}) \leq \delta$ imply $m \geq m_0$. Set $A := \{a \in \mathbb{N} : d(a, \bar{a}) \leq \delta\}$. We claim that it is enough to choose $U = \{j \in I : j_k = i_k \ (1 \leq k < n_1), j_{n_1} \in A\}$. Note first that the fixpoint of $w_a$ lies in $V_a = w_a(O)$ for all $a \in \mathbb{N}$. So, since $|V_a| \leq \varepsilon/3$ and $d(a, \bar{a}) \leq \varepsilon/3$ for $a \in A$, the union $\cup_{a \in A} V_a$ is a subset of $B_\varepsilon(\bar{a})$. Finally, $\bar{a} = w_x(\bar{a}) = \pi(k)$ for any $k$ starting with $\bar{a}$. Consider $k = i_{n_1} i_{n_1+1} \ldots$. Then, $w_{j_{n_1}-1}(\bar{a}) = w_{j_{n_1}}(\bar{a}) = \pi(j)$ and we have $\pi(U) = w_{j_{n_1}-1}(\cup_{a \in A} w_a(K)) \subset w_{j_{n_1}-1}(B_\varepsilon(\bar{a})) = B_\varepsilon(\pi(j))$ with $\varepsilon' = \lambda_{j_{n_1}-1} \varepsilon \leq \varepsilon$. \hfill $\diamond$

## 3 The invariant measure

Let $\{p_i\}_{i \in \mathbb{N}}$ be a probability vector, i.e. $0 < p_i$ and $\sum_{i \in \mathbb{N}} p_i = 1$. Due to Kolmogorov's consistency theorem [Pth, p 144] there is a probability measure $\hat{\mu}$ on $I$ corresponding to the measures $\sum_{n \in \mathbb{N}} p_n \delta_{\sigma_n}$ ($\delta_x$ is the Dirac measure concentrated in $x$) on the factors $\mathbb{N}$ of $I$. More precisely,

$$\hat{\mu}[\{i \in I : i_{k_1} = j_1, \ldots, i_{k_m} = j_m\}] = p_{j_{k_1} \ldots j_{k_m}} \quad (5)$$

for all $j, k \in I$ and $m \in \mathbb{N}$. Due to its product structure (5), $\hat{\mu}$ is invariant under the shift operators $\sigma_a : I \mapsto a_{i_1} i_2 \ldots$ of $I$:

$$\hat{\mu} = \sum_{a \in \mathbb{N}} p_a \hat{\mu}(\sigma_a^{-1}(\cdot)) = \sum_{i=1}^{\infty} p_i \hat{\mu}(\sigma_i^{-1}(\cdot)). \quad (6)$$

Provided $\pi$ is continuous

$$\mu := \hat{\mu}(\pi^{-1}(\cdot)) \quad (7)$$

is a Borel measure, more precisely a probability measure, and (6) translates immediately into (1). Moreover, $K = \pi(I)$ is the support of $\mu$ and the shift invariance $I = \cup_{a \in \mathbb{N}} \{a\} \times I$ yields

$$K = \bigcup_{a \in \mathbb{N}} w_a(K) = \bigcup_{i=1}^{\infty} w_i(K),$$
hence (2).

**Proof:** The first equation is immediate. Since $\pi$ is continuous $K$ is compact and the second ‘$\subset$’ follows. For $a \in \mathbb{N}^c$ we have $w_\pi(K) = \{a\} \subset \mathbb{N} = \{a_i\}_{i \in \mathbb{N}}$. Together with $a_i = \pi(i\ldots) = \pi(i^i \ldots) \in w_\pi(K)$ this implies the second ‘$\subset$’. $\diamond$

**Lemma 2** Assume that the SOSC holds. Then $\mu(V_{j,n}) = p_j$ for all $j \in I$ and all $n \in \mathbb{N}$.

**Proof:** First note that it is enough to prove $\mu(O) = 1$. Indeed,

$$\mu(w_j(O)) = \sum_{k=1}^{\infty} p_k \mu(w_k^{-1}(w_j(O))) = p_j \mu(O) = p_j,$$

since $\text{supp} (\mu) \subset O$ and $w_k^{-1}(w_j(O)) \cap O = \emptyset (k \neq j)$ due to the OSC. Moreover, given $i \in \mathbb{N}$ the sets $V_i$ and $w_j(O)$ $(j \in \mathbb{N}, j \neq i)$ are mutually disjoint again by the OSC. So,

$$1 = \mu(\mathbb{R}^d) \geq \mu(V_i) + \sum_{j \neq i} p_j = \mu(V_i) + (1 - p_i).$$

With $\mu(V_i) \geq \mu(\{j : j_1 = i\}) = p_i$ we obtain equality. The same argument applies to words $i_1 \ldots i_n$ of length $n$ instead of $i$. To show $\mu(O) = 1$ pick a point $x = \pi(\hat{1})$ in $O$. In case that $\hat{1}$ contains letters from $\mathbb{N}^c$ let $i_{n+1} = a \in \mathbb{N}^c$ denote the first one. Then, $x = w_{\hat{1},n}(a)$ and by definition of $\mathbb{N}^c$ there is $m \in \mathbb{N}$ such that the open set $O$ also contains $w_{\hat{1},n}(a_m) = \pi(i_1 \ldots i_{n}, m, m, m, \ldots)$. So, we always find $\hat{1} \in \mathbb{N}^c$ with $\pi(\hat{1}) \in O$. Since $O$ is open, there is $n \in \mathbb{N}$ such that $V_{j,n} \subset O$. By switching to the iterated family $w_{i_1 \ldots i_n} (i_1 \ldots i_n \in \mathbb{N}^c)$ we can assume that $n = 1$. For simplicity we assume $i_1 = 1$. In particular, $V_1 \subset O$ and $\mu(O) \geq \mu(V_1) \geq p_1 > 0$. Since $O$ is a cell we have $V_{j,1} = w_j(V_1) \subset w_j(O) \subset O$. From $w_j(O) \cap V_1 = \emptyset$ $(j \neq 1)$ and $\mu(V_{j,1}) \geq p_{j,1} = p_j p_1$ we conclude

$$\mu(O) \geq p_1 + \sum_{j \neq 1} p_j p_1 = p_1 + p_1(1 - p_1).$$

Similar, $V_{k,1} \subset w_{k,1} \subset w_k(O) \subset O$ and $w_{k,1}(O) \cap V_1 = \emptyset (k \neq 1)$. So, $V_{k,1}$, $V_{j,1}$ and $V_1$ are all disjoint provided $k, j \neq 1$. Hence,

$$\mu(O) \geq p_1 + p_1(1 - p_1) + \sum_{k,j \neq 1} p_k p_j p_1 = p_1 + p_1(1 - p_1) + p_1(1 - p_1)^2.$$ 

By induction, $\mu(O) \geq p_1 \sum_{n \in \mathbb{N}} (1 - p_1)^n = 1$ which was to show. $\diamond$

For convenience we focus on the following class of measures:
**Definition 3** We will call $\mu$ an infinite self-similar measure if it is defined by (7), $\{w_i\}_{i \in \mathbb{N}}$ being a family of similarities with $|w_i(x) - w_i(y)|/|x - y| \equiv \lambda_i \in (0, 1)$ and such that the SOSC holds.

A straightforward generalization of the fixpoint argument given by Hutchinson [Hut, p 733] shows that (1) and (2) define $\mu$ and $K$ uniquely.

### 4 Upper bound

Our results concern the sets

$$K_\alpha := \{ x = \pi(\hat{l}) : \frac{\log \mu(V_{\hat{j}n})}{\log |V_{\hat{j}n}|} \text{ is defined for all } n \text{ and tends to } \alpha \}, \quad (8)$$

where $|E|$ denotes the diameter of the set $E$. The limit $\alpha = \alpha_\alpha$ involved is called the Hölder exponent of $\mu$ at $x = \pi(\hat{l})$ and depends on the address $\hat{l}$. As with the examples in the last section points with multiple addresses can often be disregarded. The definition implies that we disregard the set $E$ of all addresses containing any letter from $\mathbb{N}$ and focus on the set of points with ‘finite addresses’ $\pi(F)$, where

$$F := \mathbb{N}^\mathbb{N} = I \setminus E.$$

Note, that $E$ is a $\mu$-nullset since $\mathbb{N}$ is a compact subset of $\mathbb{N}$ of zero measure in each factor. The corresponding points require a special treatment. In the example considered here, however, this turns out to be trivial. Moreover, although $E$ is always uncountable, $\pi(E)$ is certainly countable if $\mathbb{N}$ is. Disregarding such a set does not affect Hausdorff dimensions [F2]. Finally, it is impossible to observe infinitely many maps in simulations but rather some finite approximations like $\mu_M$ defined below and their behaviour with increasing $M$. For these measures $\mu_M$ the usual definition of Hölder exponent coincides with ours [CM] [R, th. 18 iii].

An upper bound of $d_{\text{HD}}(K_\alpha)$ is easy to obtain. Let

$$B_\alpha := \pi(\{ \hat{l} \in I \setminus E : \limsup_{n \to \infty} (\log \mu(V_{\hat{j}n})/\log |V_{\hat{j}n}|) \leq \alpha \})$$

$$C_\alpha := \pi(\{ \hat{l} \in I \setminus E : \liminf_{n \to \infty} (\log \mu(V_{\hat{j}n})/\log |V_{\hat{j}n}|) \geq \alpha \})$$

**Proposition 4** For any infinite self-similar measure we have

$$d_{\text{HD}}(B_\alpha) \leq \inf\{ \tau(q) + q\alpha : q \geq 0 \}, \quad d_{\text{HD}}(C_\alpha) \leq \inf\{ \tau(q) + q\alpha : q \leq 0 \}.$$  

Moreover, $d_{\text{HD}}(K) \leq \tau(0)$. 
Remark: The same result holds if we barely impose $|w_i(x) - w_i(y)| \leq \lambda_i|x - y|$.

Proof: For simplicity assume that the cell $O$ has diameter $|O| = 1$, hence $|V_{j_n}| = \lambda_{j_n}$. The SOSC implies $\tilde{\lambda} := \max\{\lambda_i : i \in \mathbb{N}\} < 1$ and $\mu(V_{j_n}) = p_{j_n}$. Take $q \in \mathbb{R}$. If $\tau(q) = \infty$ there is nothing to show. Otherwise take $\varepsilon > 0$. Let

$$L_N := \pi(\{i \in I \setminus E : p_{j_n} \geq \lambda_{j_n}^{q+(\alpha+\varepsilon)} \text{ for all } n \geq N\})$$

First we estimate $d_{\text{HD}}(L_N)$. For $n \geq N$ let $A_n = \{j = j_1, \ldots, j_n \in \mathbb{N}^n : p_{j_n} \geq \lambda_{j_n}^{q+(\alpha+\varepsilon)}\}$. We have

$$1 = \left(\sum_{i \in \mathbb{N}} p_i \lambda_i^{-\tau(i)}\right)^n \geq \sum_{j \in A_n} p_j \lambda_j^{-\tau(j)} \geq \sum_{j \in A_n} \lambda_j^{\alpha + \varepsilon + \tau(j)} = \sum_{j \in A_n} (V_{j_n})^{\alpha + \varepsilon + \tau(j)}.$$

Since $(V_{j_n})_{j \in A_n}$ provides a covering of $L_N$ with $|V_{j_n}| \leq \tilde{\lambda}^n$, this implies $d_{\text{HD}}(L_N) \leq q\alpha + q\varepsilon + \tau(q)$ for all $N$. The union $\bigcup_{N \in \mathbb{N}} L_N$ contains $B_\alpha$ if $q \geq 0$, resp. $C_\alpha$ if $q \leq 0$. Recalling that $d_{\text{HD}}\left(\bigcup_{N \in \mathbb{N}} L_N\right) = \inf_{N \in \mathbb{N}} d_{\text{HD}}(L_N)$ [F2] and then using that $\varepsilon > 0$ was arbitrary the first two estimates follow. Finally, for $q = 0$ the collection $(V_{j_n})_{j \in A_n}$ covers the whole support $K$ of $\mu$. This completes the proof.

\[\diamond\]

Corollary 5 For any infinite self-similar measure we have

$$d_{\text{HD}}(K_\alpha) \leq \inf\{\tau(q) + q\alpha : q \in \mathbb{R}\}.$$

Proof: $K_\alpha \subset B_\alpha \cap C_\alpha$.

\[\diamond\]

5 Separation condition

To estimate $d_{\text{HD}}(K_\alpha)$ from below we consider ‘finite approximations’ of $\mu$, i.e. self-similar measures $\mu_M$ invariant under the first $M$ maps $w_1, \ldots, w_M$. In a certain sense, the spectra $f_M$ of $\mu_M$ approximate $f$ from below which is again very helpful from the point of view of application. Fix a natural number $M$ and let $\mu_M$ be the unique self-similar measure invariant under the maps $(w_1, \ldots, w_M)$ and the probabilities $(p_{1}^{(M)}, \ldots, p_{M}^{(M)})$ where $p_k^{(M)} = p_k / N_M$ and $N_M := \sum_{i=1}^{M} p_i$. Hence

$$\mu_M = \sum_{i=1}^{M} p_i^{(M)} \mu_M(w_i^{-1}(\cdot)).$$
Assuming that $w_1, \ldots, w_M$ satisfies the DOSC, Cawley et Mauldin [CM] showed that the multifractal formalism holds for $\mu_M$. This means that the spectrum $f_M(\alpha)$ equals the Legendre transform $f_M(\alpha)$ of the singularity exponents $\tau_M(q)$ which are given by
\[
\sum_{i=1}^{M} (p_i^{(M)})^q \lambda_i^{\tau_M(q)} = 1.
\]
In particular,
\[
d_{\text{HD}}(K_i^{(M)}) = f_M(\alpha) = \tau_M(q) - q\tau_M'(q)
\]
for $\alpha = -\tau_M'(q)$. (A short proof and the spectrum obtained by box counting are given in [R].) The separation condition we require is therefore the following one.

**Definition 6 (Finite Separation Condition)** A family $\{w_i\}_{i \in \mathbb{N}}$ and its self-similar measure $\mu$ is called finitely separated if for all $M \in \mathbb{N}$ there is a bounded open set $O_M$ such that the sets $w_i(O_M) (i = 1, \ldots, M)$ are mutually disjoint subsets of $\overline{O_M}$, i.e. $(w_1, \ldots, w_M)$ satisfies the DOSC.

Of course, a family $\{w_i\}_{i \in \mathbb{N}}$ satisfying the DOSC is finitely separated, but not vice versa. Even connected supports $K$ are possible, provided the copies $w_i(K)$ accumulate ‘nicely’ at the boundary of a cell:

**Proposition 7** Let $\{w_i\}_{i \in \mathbb{N}}$ be a family of similarities on the line, i.e. $w_i(x) = \theta_i \lambda_i x + u_i$ with $\theta_i \in \{-1, 1\}$. Assume that the OSC holds with some open interval $O = [u, v]$. If $\mathbb{N}^c$ contains both boundary points of $O$ then $\{w_i\}_{i \in \mathbb{N}}$ is finitely separated. The same is true if all $\theta_i$ are equal and at least one boundary point of $O$ lies in $\mathbb{N}^c$.

**Proof:** The basic idea is to shrink $O$ to become a smaller open set $O_M$. First, consider the case $u \in \mathbb{N}^c$. We show that $u$ cannot lie in any $V_i = w_i([u, v])$ ($i \in \mathbb{N}$). Assume the contrary. This implies that $V_i = [u, u + \lambda_i (v - u)]$. By definition of $\mathbb{N}^c$ there is $k \in \mathbb{N}$ such that $|u - a_k| \leq \lambda_i (v - u)/2$. Since $a_k \in V_k \subset [u, v]$ the fixpoint $a_k$ must lie in the interior of $V_i$ which contradicts the OSC. Now take any $M \in \mathbb{N}$. Since no $V_i$ contains $u$ there is $\varepsilon = \varepsilon(M)$ such that $V_i \subset O_M := [u + \varepsilon, v] \subset O$ for $i = 1, \ldots, M$. Since $w_i(O_M) \subset V_i$, $O_M$ satisfies the OSC for $w_1, \ldots, w_M$. Finally, the OSC for $O$ and the equal signs of the $\theta_i$ imply that the sets $w_i(O_M)$ ($i = 1, \ldots, M$) are mutually disjoint. This was to show. In case $v \in \mathbb{N}^c$, a similar argument applies to some $O_M := [u, v - \varepsilon]$. If $u$ and $v$ are both contained in $\mathbb{N}^c$ then we can choose $O_M := [u + \varepsilon, v - \varepsilon]$ and the signs of the $\theta_i$ are not important. \hfill \Box
6 Lower bound

Now we proceed to estimate \( d_{\text{HD}}(K_\alpha) \). We recall some details of the proof of (9) which are useful here. Fix \( q \) and set \( \overline{p}_k := (p_k(M))^{\frac{1}{2}}(\lambda_k)^{\tau_M(q)} \). The main idea is to focus on the words \( \overline{i} \) which produce the desired Hölder exponent \( \alpha = \alpha_M(q) \) according to (9)

\[
\alpha_M(q) := \frac{\sum_{i=1}^{M} \log(p_i(M))(p_i(M))^{\frac{1}{2}}(\lambda_i)^{\tau_M(q)}}{\sum_{i=1}^{M} \log(\lambda_i)(p_i(M))^{\frac{1}{2}}(\lambda_i)^{\tau_M(q)}} = \frac{\sum_{i=1}^{M} \log(p_i(M))\overline{p}_i}{\sum_{i=1}^{M} \log(\lambda_i)\overline{p}_i} = -\tau'_M(q).
\]

As in (5), let \( \hat{\mu}_{q,M} \) be the product measure on the product space \( I_M := \{1, \ldots, M\}^\mathbb{R} \) corresponding to the factor measures \( \overline{p}_i = \overline{p} \overline{p}_M \). The tool to observe the Hölder exponent are the random variables \( X_n := \log p_n(M) \) and \( Y_n := \log \lambda_n \) which are i.i.d. due to the product structure (5) of \( \hat{\mu}_{q,M} \). Consider the set \( K_{\alpha}(M) \) of all words \( \overline{i} \in \{1, \ldots, M\}^\mathbb{R} \) such that simultaneously

\[
\frac{1}{n} \log p_n(M) = \frac{1}{n} \sum_{k=1}^{n} X_k \rightarrow E[X_n] = \sum_{i=1}^{M} \log(p_i(M))\overline{p}_i, \quad (10)
\]

and \( (1/n) \log \lambda_n \rightarrow E[Y_n] = \sum_{i=1}^{M} \log(\lambda_i)\overline{p}_i \). Obviously, \( K_{\alpha}(M) = \pi(K_{\alpha}(M)) \).

By the Law of Large Numbers (10) is a \( \hat{\mu}_{q,M} \)-almost sure event. Hence, \( \hat{\mu}_{q,M}(K_{\alpha}(M)) = 1 \) and Frostman’s lemma [F2, pr. 4.9] yields \( d_{\text{HD}}(K_{\alpha}(M)) \). For \( \mu \)

we conclude that

\[
\frac{\log \mu(V_{\overline{j}_n})}{\log |V_{\overline{j}_n}|} = \frac{\log p_n}{\log \lambda_n} \rightarrow \beta_M(q) := \alpha_M(q) + \frac{\log(N_M)}{\sum_{i=1}^{M} \log(\lambda_i)(p_i(M))^{\frac{1}{2}}(\lambda_i)^{\tau_M(q)}} \quad (n \rightarrow \infty)
\]

for every \( \overline{j} \in K_{\alpha}(M) \). Hence \( K_{\alpha}(M) \subseteq K_{\beta_M(q)} \) and

\[
d_{\text{HD}}(K_{\beta_M(q)}) \geq \tau_M(q) - q\tau_M'(q).
\]

It remains to consider the limit \( M \rightarrow \infty \). Let

\[
\mathbb{I} := \{q \in \mathbb{R} : \tau \text{ is finite in a neighbourhood of } q\}.
\]

Lemma 8 For all \( q \) with finite \( \tau(q) \) we have \( \tau_M(q) \rightarrow \tau(q) \) as \( M \rightarrow \infty \).

Proof: Assume there is \( \gamma > 0 \) and a sequence of integers \( (M_n)_{n \in \mathbb{N}} \) such that \( \tau_{M_n}(q) \geq \tau(q) + \gamma \) for all \( n \). Then, using monotonicity, (3), \( N_M \rightarrow 1 \) and (9) we obtain

\[
1 > \lim_{n \rightarrow \infty} \sum_{i=1}^{M_n} p_i^\gamma \lambda_i^{\tau(q)+\gamma} \geq \lim_{n \rightarrow \infty} (N_M)^\gamma \sum_{i=1}^{M_n} (p_i(M_n))^\gamma \lambda_i^{\tau_M(q)} = 1,
\]

which contradicts the assumption. Hence \( \tau_M(q) \rightarrow \tau(q) \) as \( M \rightarrow \infty \).
a contradiction. A similar argument shows that \( \liminf_{M \to \infty} \tau_M(q) \geq \tau(q) \).

\[ \diamond \]

**Lemma 9** Let \( \mu \) be an infinite self-similar measure. Assume there are numbers \( r, R \) such that \(-\infty < \log r \leq (1/n) \log p_n \leq \log R < 0 \ \forall n \). Then, we distinguish two cases:

a) (left sided) If \( \lim_{n \to \infty} 1/n \log \lambda_n = 0 \), then \( \mathbb{I} = (0, \infty) \).

b) (right sided) If \( \lim_{n \to \infty} 1/n \log \lambda_n = -\infty \), then \( \mathbb{I} = (-\infty, 1) \).

In both cases

\[ \lim_{M \to \infty} \beta_M(q) = \lim_{M \to \infty} \alpha_M(q) = \alpha(q) := \frac{\sum_{i \in \mathbb{N}} \log(p_i) p_i \lambda_i^{(q)}}{\sum_{i \in \mathbb{N}} \log(\lambda_i) p_i \lambda_i^{(q)}} \tag{13} \]

locally uniform in \( \mathbb{I} \). In particular, \( \tau'(q) = -\alpha(q) \) for \( q \in \mathbb{I} \). Moreover, all statements remain true if the sequences \( \{p_i\}_{i \in \mathbb{N}} \) and \( \{\lambda_i\}_{i \in \mathbb{N}} \) are exchanged.

**Remark:** The case a) is called left sided, since \( \tau(q) \) is not defined for \( q < 0 \) and its Legendre transform \( f(\alpha) \) has only an increasing (=left) part. If \((1/n) \log p_n \) and \((1/n) \log \lambda_n \) are both bounded away from \(-\infty \) and 0 then \( \mathbb{I} = \mathbb{R} \). In this case, a sufficient condition for (13) is that \((1/n) \log p_n \) and \((1/n) \log \lambda_n \) both converge with limits different from 0 and \(-\infty \). Then, \( f \) has the usual \( \cap \)-shape. Finally, it is easy to construct examples which have \( \mathbb{I} = (0, 1) \).

**Proof:** First, note that \( \beta_M(q) \) and \( \alpha_M(q) \) converge simultaneously with the same (uniform) limit as \( M \to \infty \). Indeed, \( N_M \to 1 \) and a mean value argument gives

\[ \left| \sum_{i=1}^{M} \log(\lambda_i) \left( p_i^{(M)} \right)^{q} \lambda_i^{\tau_M(q)} \right| \geq -\log(\max\{\lambda_i : i \in \mathbb{N}\}) > 0. \tag{14} \]

Let us compute \( \mathbb{I} \). Due to monotonicity and convexity of the exponential \( \mathbb{I} \)
must be an interval and \( \tau \) is decreasing and convex, hence continuous in \( \mathbb{I} \). We show that \((0, 1) \subset \mathbb{I}\) without further assumption. Let \( q_1 \in (0, 1) \). Due to the OSC \( \sum_{i \in \mathbb{N}} p_i^{q_1} \lambda_i^{d} < \sum_{i \in \mathbb{N}} \lambda_i^{d} \leq 1 < \sum_{i \in \mathbb{N}} p_i^{q_1} \). Also, \( h(x) := \sum_{i \in \mathbb{N}} p_i^{q_1} \lambda_i^{x} \leq \sum_{i \in \mathbb{N}} p_i^{q_1} \) converges uniformly for \( x \in [0, d] \). Hence, \( h \) is continuous. Since it is strictly decreasing there must be a (unique) solution \( \tau(q_1) \) of (3) which must lie in \([0, d]\).

Now, assume that \((1/n) \log \lambda_n \to 0 \). Let \( q_2 > 1 \). There is \( x_0 < 0 \) such that \( p_1^{q_2} \lambda_1^{x_0} > 1 \). Choose \( s \in (0, 1) \) such that \( \gamma := R^{q_2} s^{x_0} < 1 \). By assumption
there is $i_0(s)$ such that $\lambda_i > s^i$ for $i \geq i_0$. So, $\sum_{i \in \mathbb{N}} p_i^{\alpha q} \lambda_i^{x_0}$ converges to a number $> 1$. Since $\sum_{i \in \mathbb{N}} p_i^{\alpha q} < 1$, there must be a solution $\tau(q)$ of (3) which
must lie in $[x_0, 0]$. Due to monotonicity, (3) has a solution $\tau(q)$ which
must lie in $[x_0, d]$ for all $q \in [q_1, q_2]$. Similar estimates show that (3) has no solution for
$q < 0$. Hence, $\mathbb{D} = (0, \infty)$. For further use note that for every $q \in [q_1, q_2]$ and
every $i \geq i_0(s)$
\[ p_i^{\alpha q} \lambda_i^{\tau(q)} \leq p_i^{\alpha q} \lambda_i^{x_0} \leq (R^{\alpha q} s_0^x)^i = \gamma^i. \]

In order to show $\beta_M(q) \to \alpha(q)$ it is enough to prove that
\[ \sum_{i=1}^{M} \log(x_i) (p_i^{(M)})^{\alpha q} \lambda_i^{\tau_M(q)} - \sum_{i \in \mathbb{N}} \log(x_i) p_i^{\alpha q} \lambda_i^{\tau(q)} \to 0 \quad (M \to \infty) \quad (15) \]
for $\{x_i\}_{i \in \mathbb{N}} = \{p_i\}_{i \in \mathbb{N}}$ and for $\{x_i\}_{i \in \mathbb{N}} = \{\lambda_i\}_{i \in \mathbb{N}}$. We split the difference (15)
into two parts $A + B$. A mean value argument yields
\[ A := \sum_{i=1}^{M} \log(1/x_i) \bar{p}_i - \sum_{i=1}^{M} \log(1/x_i) p_i^{\alpha q} \lambda_i^{\tau(q)} = \log(1/\xi_M) \sum_{i=1}^{M} \bar{p}_i - p_i^{\alpha q} \lambda_i^{\tau(q)} \]
with $\min\{x_i : 1 \leq i \leq M\} \leq \xi_M \leq \max\{x_i : 1 \leq i \leq M\}$. Since $\sum_{i=1}^{M} \bar{p}_i = 1$,
we have
\[ A = \log(1/\xi_M) \sum_{i=M+1}^{\infty} p_i^{\alpha q} \lambda_i^{\tau(q)}. \]

For both choices of $\{x_i\}_{i \in \mathbb{N}}$ we must have $\xi_M \geq r^M$ for large $M$. Hence,
\[ 0 \leq A \leq M \log(1/r) \cdot \sum_{i=M+1}^{\infty} \gamma^i \to 0 \quad (M \to \infty) \]

The rest of the difference (15) can be treated with similar arguments:
\[ B := \sum_{i=M+1}^{\infty} \log(1/x_i) p_i^{\alpha q} \lambda_i^{\tau(q)} \leq \sum_{i=M+1}^{\infty} i \log(1/r) \gamma^i \to 0 \quad (M \to \infty). \]

This proves that (15) holds uniformly for $q \in [q_1, q_2]$ which implies (13). In
fact, using (14) and (11) one obtains that $\tau^{(M)}_q$ converges uniformly to $-\alpha(q)$
in every compact subset of $\mathbb{D}$.

Next, assume $(1/n) \log \lambda_n \to -\infty$. Again, straightforward estimates yield $\mathbb{D}$.
The locally uniform convergence of (15) can be proven in a similar manner as
above, but with one adoption. In this case the `strong’ sequence is $\{\lambda_i\}_{i \in \mathbb{N}}$
the rate of convergence of which is not known. But we may assume that it
is monotonously decreasing since reordering changes neither the geometry
nor the value of positive sums. Fix $q_3 < 0$ and choose $S \in (0,1)$ such that
\[ \gamma := r^{33} S^{\tau(q_1)/2} < 1. \]
Recall that $\tau(q_1) > 0$. By assumption $\lambda_i \leq S^i$ for $i \geq i_1(S)$. For both choices of $\{x_i\}_{i \in \mathbb{N}}$, the estimate
\[ \log(1/\xi_M) \leq \log(1/\lambda_M) \leq \log(1/\lambda_i) \leq \lambda_i^{-\tau(q_1)/2} \]
holds for $M \geq M_0$, $M_0$ depending on $q_1$ and on $\{p_i\}_{i \in \mathbb{N}}$, and for $i \geq M$. For $q \in [q_3, q_1]$ we obtain
\[ \log(1/\xi_M) \sum_{i=M+1}^{\infty} p_i^q \lambda_i^{\tau(q)} \leq \sum_{i=M+1}^{\infty} p_i^{q_3} \lambda_i^{\tau(q_1)-\tau(q_1)/2} \leq \sum_{i=M+1}^{\infty} (r^{33} S^{\tau(q_1)/2})^i = \sum_{i=M+1}^{\infty} \gamma^i \]
provided $M \geq \max(M_0, i_1)$. It is now easy to complete the proof of b).

Finally, exchanging the sequences $\{p_i\}_{i \in \mathbb{N}}$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ means to exchange $\tau$ and $q$, hence to switch the left and right sided case. Indeed, the additional assumption on the asymptotic behaviour of the sequences were only used to prove (15) which is symmetric in $p$ and $\lambda$, so to say. The only break of symmetry is $\sum_{i \in \mathbb{N}} p_i = 1$ while $\sum_{i \in \mathbb{N}} \lambda_i^2 \leq 1$. So, precisely speaking, $\tau$ becomes $q \cdot d$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ has to be ‘normalized’ by multiplication with a suitable constant. Then, the same argumentation applies to give the obvious results.

\[ \odot \]

7 Main result

Theorem 10 (Multifractal Formalism) Let $\mu$ be a finitely separated infinite self-similar measure.\(^2\) Assume there are numbers $r, R$ such that $-\infty < \log r \leq (1/n) \log p_n \leq \log R < 0 \forall n$. Furthermore, assume that $\lim_{n \to \infty} (1/n) \log \lambda_n$ exists and is either zero or $-\infty$. Then, for $\alpha = \alpha(q) = -\tau'(q)$ ($q \in \mathbb{H}$),
\[ d_{\text{HD}}(K_\alpha) = f(\alpha) := \inf \{ \tau(q) + qa : q \in \mathbb{R} \}. \]
The same holds if there are numbers $s, S$ such that $-\infty < \log s \leq (1/n) \log p_n \leq \log S < 0 \forall n$ and $\lim_{n \to \infty} (1/n) \log p_n$ is either zero or $-\infty$.

Remark: Together with proposition 4 we get the Hausdorff dimensions of $B_\alpha$ and $C_\alpha$ at $\alpha = \alpha(q)$ for $q \geq 0$ and $q \leq 0$ respectively.

To give a more precise statement let us agree on the notation $d_{\text{HD}}(\emptyset) = -\infty$.

\(^2\)According to the recent result of Arbeiter and Patzschke [AP] the finitely separation condition can be replaced with the weaker OSC.
Corollary 11 Let \( \mu \) be a finitely separated infinite self-similar measure. Assume there are numbers \( r, R \) such that \(-\infty < \log r \leq (1/n) \log p_n \leq \log R < 0 \) \( \forall n \).

Then,

a) left sided If \( \lim_{n \to \infty} (1/n) \log \lambda_n = 0 \) then

\[
d_{\text{HD}}(K_\alpha) = \begin{cases} 
\tau(0) = d_{\text{HD}}(K) & \text{if } \alpha \geq \alpha_0 \\
\tau(q) - q^\tau(q) & \text{if } \alpha_0 < \alpha = \alpha(q) > \alpha(\infty) \\
D_{\infty} & \text{if } \alpha = \alpha(\infty) \\
-\infty & \text{if } \alpha < \alpha(\infty)
\end{cases}
\]

where \( \alpha_0 := \alpha(0) = -\tau'(0), \alpha(\infty) := \lim_{q \to \infty} \alpha(q) = \inf \{ \alpha(q) : q \in \mathbb{R} \} \) and \( D_{\infty} \) is defined through \( \sum q_i^{D_{\infty}} = 1 \), the sum \( \sum q_i \) running over all \( i \) with \( \log p_i/\log \lambda_i = \alpha(\infty) \). Note that \( \alpha_0 \) may be infinite.

b) right sided If \( \lim_{n \to \infty} (1/n) \log \lambda_n = -\infty \) then

\[
d_{\text{HD}}(K_\alpha) = \begin{cases} 
\alpha & \text{if } 0 \leq \alpha \leq \alpha_1 \\
\tau(q) - q^\tau(q) & \text{if } \alpha_1 < \alpha = \alpha(q)\alpha(-\infty) \\
D_{-\infty} & \text{if } \alpha = \alpha(-\infty) \\
-\infty & \text{if } \alpha > \alpha(-\infty)
\end{cases}
\]

where \( \alpha_1 := \alpha(-1) = -\tau'(1) \), \( \alpha(-\infty) := \lim_{q \to -\infty} \alpha(q) = \sup \{ \alpha(q) : q \in \mathbb{R} \} \) and \( D_{-\infty} \) is defined through \( \sum q_i^{D_{-\infty}} = 1 \), the sum \( \sum q_i \) running over all \( i \) with \( \log p_i/\log \lambda_i = \alpha(-\infty) \). Note that \( \alpha_1 \) may vanish.

Remark: For any left sided infinite self-similar measure \( \mu \) on the line with \( \sum \lambda_i = 1 \) there is a right sided ‘inverse’ measure \( \mu^* \) obtained by exchanging probabilities \( p_i \) and ratios \( \lambda_i \). Inverse measures can be defined in more general situations which will be discussed in a forthcoming paper [MR].

Proof: Case a): Since \( A_i := \log p_i/\log \lambda_i \to \infty \) as \( i \to \infty \) it must reach its minimum, say \( A_k \). Obviously, \( A_k \) is the smallest Hölder exponent under consideration here (8). It is known [CM, R] that \( \alpha_M(\infty) = \min \{ \log p_i/\log \lambda_i : 1 \leq i \leq M \} = \inf \{ \alpha_M(q) : q \in \mathbb{R} \} \) is the minimal Hölder exponent of \( \mu_M \). Consequently, \( \alpha_M(\infty) = A_k \) for \( M \geq k \). This shows that \( \alpha(\infty) = A_k \) exists trivially and also that \( K_a = \emptyset \) for \( \alpha < \alpha(\infty) \).

It is clear that only words \( j_1 \) with \( \log p_{n_0(i)}/\log \lambda_{n_0(i)} = A_k \) for all \( n \geq n_0(i) \) have a Hölder exponent \( \alpha(\infty) \). Moreover, there is an integer \( l \) such that \( A_i > \alpha(\infty) \) for all \( i > l \). Hence, \( K_{\alpha(\infty)} \cup \{ \omega_{j_1 \ldots j_n}(K_{\alpha(\infty)}) \} \) with the union \( \cup \) running over all finite sequences \( j_1 \ldots j_n \) of integers. Recall that \( \alpha_l(\infty) = \alpha(\infty) \). In particular,
\( \cup' \) is countable. It is known [R] that \( d_{\text{hd}}(K_{\gamma_i(l)}) = f_i(\alpha_i(\infty)+) = \gamma_i \), where 
\[ \sum' \lambda_i \gamma_i = 1 \text{ with } \sum' \text{ running over all } i \leq l \text{ with } A_i = \alpha_i(\infty). \]
Since any set \( E \) has the same dimension as \( w_i(E) \) and since \( d_{\text{hd}}(\cup_{\mathbb{N}})E_n = \sup_{\mathbb{N}} d_{\text{hd}}(E_n) \) for any countable collection of sets \( E_n \) [F2], we obtain \( d_{\text{hd}}(K_{\gamma_i(l)}) = \gamma_i \). Note also, that we can choose \( l \) as large as we wish without changing the result. This shows that the definition of \( D_\infty \) makes sense and that \( D_\infty = \gamma_i \). In particular, \( d_{\text{hd}}(K_{\gamma_i(l)}) = f(\alpha(\infty)+) = \gamma_i \) for every large enough \( l \).

To compute the limit of the left hand side of (12) we have to use the continuity of \( f \) which is guaranteed by lemma 9. Let \( \beta = \alpha(q) \) with \( q \in \mathbb{D} \). Recall that \( \mathbb{D} \) is an interval and that \( \alpha(\cdot) \) is continuous and monotonous. Hence, there are \( q_1 \) and \( q_2 \) such that \( \alpha(q_1) \leq \beta < \alpha(q_2) \). From lemma 9 we conclude that there is \( M_0 \) such that \( \alpha_M(q)< \beta < \alpha_M(q) \) for all \( M \geq M_0 \). Due to concavity of \( f_M \) and due to (12) \( d_{\text{hd}}(K_{\beta}) \geq \min\{f_M(\alpha_M(q_1)), f_M(\alpha_M(q_2))\}. \) Letting \( M \to \infty \), lemmata 8 and 9 give \( d_{\text{hd}}(K_{\beta}) \geq \min\{f(\alpha(q_1)), f(\alpha(q_2))\} \). Letting \( (q_2 - q_1) \to 0 \) we obtain \( d_{\text{hd}}(K_{\beta}) \geq f(\beta) \) by continuity of \( f \). Proposition 4 gives equality.

The well known fact \( d_{\text{hd}}(\text{supp}(\mu_M)) = \tau_M(0) \) and corollary 5 imply \( d_{\text{hd}}(K) = \tau(0) \).

If \( \alpha_0 = \infty \) there is nothing more to show. Otherwise, i.e. if \( \alpha_0 < \infty \), note that \( \alpha_M(\infty) = \lim_{q \to -\infty} \alpha_M(q) = \sup \{\alpha_M(q) : q \in \mathbb{R} \} = \max \{\log p_i/\log \lambda_i : 1 \leq i \leq M \} \to \infty (m \to \infty). \) From the concavity of \( f_M \) and (12) one obtains \( d_{\text{hd}}(K_{\beta}) \geq l_{M,\beta}(\alpha) \) where \( \alpha \) and \( \beta \) are related through (11), where \( l_{M,\beta} \) is the line through the points \( (\alpha_M(q), f_M(\alpha_M(q))) \) and \( (\alpha_M(\infty), 0) \) and where \( \alpha_M(q) \leq \alpha \leq \alpha_M(\infty). \) Fix \( q > 0 \) and let \( M \to \infty. \) Lemma 9 gives \( d_{\text{hd}}(K_{\beta}) \geq f(\alpha(q)) \) for \( \beta > \alpha(q). \) Finally, as \( q \downarrow 0, \alpha(q) \uparrow \alpha_0 \) and \( f(\alpha(q)) = \tau(q) - q\tau'(q) \to \tau(0) - 0 \cdot \alpha_0 = d_{\text{hd}}(K) \).

Case b) The conditions on the asymptotic behaviour of the sequences \( \{p_i\}_{i \in \mathbb{N}} \) and \( \{\lambda_i\}_{i \in \mathbb{N}} \) are only used to determine \( \mathbb{D} \) and to prove \( f_M \to f \). According to lemma 9 exchanging the two sequences means basically to exchange \( q \) and \( \tau \), hence to exchange left and right sided case. \( \diamond \)

8 Examples

Three examples of spectra are given which cannot be observed with finite self-similar measures. The first is the left sided spectrum given in [MEH]. The second one is left sided as well showing that infinite Hölder exponents can
occur for multiplicative cascades with ‘binomial’ structure (see Ex. 2). The third example gives a kind of ‘inverse’ left sided spectrum what we called right sided. It shows that Hölder exponent zero is possible without presence of atoms. This possibility has already been mentioned in [M].

**Example 1 The left sided spectrum of [MEH]** The infinite multinominal measure considered by Mandelbrot et al. [MEH] fits perfectly into our framework. Proposition 7 and theorem 10 apply with the cell $O := (0, 1)$ showing that the usual multifractal formalism holds. Let

$$w_i(x) := \frac{1}{(i+1)^\kappa} + \left(\frac{1}{i^\kappa} - \frac{1}{(i+1)^\kappa}\right)x$$

and

$$p_i := \frac{1}{2},$$

where $\kappa > 0$ is a parameter. The support of $\mu$ is $K = [0, 1]$. Some lengthy but straightforward calculations show [MEH, Appendix] that the spectrum is left sided, in particular $f(\alpha) \uparrow 1$ for $\alpha \to \infty$. It is worth noting that the rate of approach of $f(\alpha)$ towards 1 depends on $\kappa$ as indicated in Fig. 2. Set

$$\alpha_0 := -\tau'(0+) = \left\{ \begin{array}{ll}
\infty & \text{if } \kappa \leq 1, \\
\sum_{i \in \mathbb{N}} \log(p_i)\lambda_i / \sum_{i \in \mathbb{N}} \log(\lambda_i)\lambda_i & \text{otherwise}.
\end{array} \right.$$ 

As $\alpha \uparrow \alpha_0$ we have

$$f(\alpha) \approx \left\{ \begin{array}{ll}
1 - c(\alpha_0 - \alpha)^{\kappa'} & \text{if } \kappa > 1, \\
1 - c\exp(-c\alpha) & \text{if } \kappa = 1, \\
1 - c\alpha^{\kappa''} & \text{if } \kappa < 1,
\end{array} \right.$$ 

where $\kappa' = \max\{2, \kappa/(\kappa - 1)\}$ and $\kappa'' = \kappa/(\kappa - 1)$.

Let us add some remarks on the peculiar Hölder exponent $\alpha = \infty$. First of all, $\mathbb{N}' = \{0\}$. The Hölder exponent is obviously infinite there and at all the iterated images, i.e. at the boundary points $1/i^{\kappa}$ of the intervals appearing in the construction. These points are the one with multiple addresses and with ‘infinite addresses’ from $E$. Furthermore, the measure is supported by $K = [0, 1]$. Disregarding the countable number of points with multiple addresses we have

$$\frac{\log \mu(V_{j\alpha})}{\log |V_{j\alpha}|} = \frac{\log p_{j\alpha}}{\log \lambda_{j\alpha}} \to \alpha(0) = -\tau'(0+)$$

for (Lebesgue) almost every $x = \pi(j)$. This Hölder exponent $\alpha(0)$ is infinite for $\kappa \leq 1$.

The arguments given in the proof of theorem 10 and in Fig. 1 show that this behaviour can be observed also in numerical simulations since the spectrum is
determined by ‘finite approximations’. On the other hand, this ‘finite approximations’ are constituted by points with bounded addresses, i.e. with \( i_n \leq M \) for all \( n \), which lie on fractal sets of dimensions \( D_M \) strictly less than 1, in particular \( \sum_{i=1}^{M} \lambda_i^{D_M} = 1 \). Hence, these points form a Lebesgue nullset. Finally, note that
\[
\frac{\log \mu(V_{jn})}{\log |V_{jn}|} = \frac{\log p_{jn}}{\log \lambda_{jn}} \to \alpha_1 := \tau'(1) = \sum_{i \in I^M} \frac{\log(p_i)p_i}{\sum_{i \in I} \log(\lambda_i)p_i}
\]
for \( \mu \) almost every \( x = \pi(\underline{j}) \).

**Proof:** The argument is essentially the same as (10) with \( M = \infty \). Pick any \( q \geq 0 \) and let \( \hat{\mu}_q \) be the product measure on \( I \) corresponding to the probabilities \( \overline{p}_i := p_i^q \lambda_i^{-\tau(q)} \). Then, \( \mu_q := \hat{\mu}_q(\pi^{-1}(\cdot)) \) satisfies (1) with \( \overline{p}_i \) instead of \( p_i \). Consider the i.i.d. random variables \( X_n := \log p_{jn} \) and \( Y_n := \log \lambda_{jn} \). Then,
\[
\frac{\log p_{jn}}{\log \lambda_{jn}} = \frac{1}{n} \log \left( \frac{p_{jn}}{\lambda_{jn}} \right) \to \frac{E[X_n]}{E[Y_n]} = \frac{\sum_{i \in I^M} \log(p_i)p_i^{\tau(q)}}{\sum_{i \in I^M} \log(\lambda_i)p_i^{\tau(q)}} (n \to \infty)
\]
\( \hat{\mu}_q \) almost surely. Note that the denominator \( E[Y_n] \) is always finite and that the LLN holds also if the numerator \( E[X_n] \) is infinite. Since only countably many points have multiple addresses this translates into the desired \( \mu_q \)-almost sure statement. Finally, \( \mu_1 = \mu \) is obvious. For \( q = 0 \) we have \( \overline{p}_i = \lambda_i \) and the Lebesgue measure restricted to \([0,1]\) satisfies (1) with \( p_i = \lambda_i \). By uniqueness it must be \( \mu_0 \).

\[\Diamond\]
Example 2 A left sided spectrum with binomial structure. An example of a self-similar measure with geometrically decreasing ratios $\lambda_i$ and left sided spectrum is given by

$$w_n(x) = \frac{1}{2^{n+1}} x^{\frac{1}{2^{n+1}}} + \frac{1}{2^{n+1}} x^{\frac{1}{2^{n+1}}}$$

and $p_n := \rho \cdot \frac{1}{2^{n+1}}$.

where $\kappa > 0$ is a parameter and $\rho$ is the obvious normalization constant. Similar calculations as in [MEH] give exactly the same behaviour of $f(\alpha)$ as in Ex. 1. This example has been included for its transparent geometry. In particular, it shows the binomial structure as found with the binomial measure invariant under $w_1(x) = x/2$ and $w_2(x) = x/2^1 + 1/2$, however, with a different mass distribution.

Example 3 (A right sided spectrum) Exchange the ratios $\lambda_i$ and the probabilities $p_i$ of Ex. 1. Again, similar calculations as in [MEH] give the different rates of convergence of $f(\alpha) \to \alpha$ as $(\alpha \rightarrow \alpha_1)$ where $\alpha_1 := -\tau'(1+)$.

In particular, if $\kappa > 1$ then $\alpha_1$ is positive, otherwise it is zero. As $\alpha \rightarrow \alpha_1$ the difference $f(\alpha) - \alpha$ behaves like a power of $\alpha - \alpha_1$ for $\kappa > 1$ (and vanishes for $\alpha < \alpha_1$), like an exponential of $\alpha$ for $\kappa = 1$ and like a power of $\alpha$ for $\kappa < 1$.

But it is even simpler to argue in the following way: Let us denote the singularity exponents and the spectrum of Ex. 1 by $\tau^*$ and $f^*$ respectively. Then $\tau$ is simply the inverse function of $\tau^*$. In other words $\tau = q^*$ and $q = \tau^*$.

Applying the Legendre transform shows that the spectrum $f(\alpha)$ is related to $f^*(\alpha)$ by

$$f(\alpha) = \alpha f^*(1/\alpha).$$

This relation is generally valid for ‘inverse’ measures which is proven in a forthcoming paper. Here, it shows that the different rates of $f(\alpha) \to \alpha$ are directly related to the one of $f^* \to 1$. Compare Fig. 1.

In conclusion: with this measure one finds the Hölder exponent zero without presence of atoms, a possibility which has been mentioned in [M].

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