Multifractal products of stochastic processes: A preview

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Abstract

Motivated by the need for multifractal processes with stationary increments we introduce a construction of random multifractal measures based on iterative multiplication of stationary stochastic processes. We establish conditions for the $\mathcal{L}^2$-convergence and non-degeneracy of the limit process in a general setting. Proceeding then to multiplying piecewise constant processes, we proof continuity of the limit and show some other interesting properties.

1 Introduction

This study is strongly motivated by the seek of new models for teletraffic. In various recent papers (see e.g. [RLV97, LVR97, MN97, FGW98]), it has been demonstrated that teletraffic exhibits multifractal properties. There are many ways to construct random multifractal measures varying from the simple binomial measures to measures generated by branching processes (see e.g., [Man72, Man74, Fal94, AP96, Pat97, RCRB98]. In teletraffic modeling, we would like to have, in addition to simplicity of the construction, also stationarity of the increments. Unfortunately, most of the multifractal models introduced so far lack this property. Although, Jaffard has shown that Lévy processes are multifractal [Jaf96], but unfortunately (from the point of teletraffic modeling) increments of a Lévy process are, in addition to stationary, also independent. Moreover, Lévy processes have a linear multifractal spectrum while real data traffic exhibits strictly concave spectra [RLV97, LVR97, MN97, RCRB98].

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In its simplest form our model is based on the multiplication of independent rescaled stochastic processes $\Lambda^{(i)}(\cdot) \overset{d}{=} \Lambda(b^i \cdot)$ which are piecewise constant. It is instructive to compare it to a Fourier decomposition where one represents or constructs a process by superposition of oscillations $\sin(\lambda_i t)$. In multiplying rather than adding rescaled versions of a 'mother' process we obtain a process with novel properties which are best understood not in an additive analysis, but in a multiplicative one.

Moreover, processes emerging from multiplicative construction schemes have positive increments and exhibit typically a 'spiky' appearance. The so-called multifractal analysis describes the local structure of a process in terms of scaling *exponents*, accounting for (being adapted to) the multiplicative structure.

With our scheme we generalize the construction of the binomial cascade[1] in randomizing it in a natural and stationary way. As with the cascades, an infinite product of random processes will typically be zero, and one has to take a distributional limit rather than pointwise limit. In more simple words, a multiplier $\Lambda^{(i)}(t)$ should not be evaluated in points but should be seen as redistributing or re-partitioning mass. Again in other words, $\Lambda^{(i)}(t)$ can be thought of as a local change in the arrival rate where one is interested actually in the integrated 'total load' process. Consequently, we set

$$A_n(t) = \int_0^t \Lambda_n(s)\ ds = \int_0^t \prod_{i=0}^{n} \Lambda^{(i)}(s)\ ds.$$

This paper is a preview of [MNR99]. First, a construction of multifractal measures based on iterative multiplication of stationary stochastic processes is introduced. After that the case of piecewise constant processes with exponentially distributed sojourn times is analyzed. Finally, a glance of the related multifractal analysis is given.

2 Construction

Let us consider a family of independent positive stationary processes $\{\Lambda^{(i)}(t)\}$ with

$$\begin{aligned} \mathbf{E}\Lambda^{(i)}(t) &= 1, && (1)\\ \operatorname{cov}(\Lambda^{(i)}(t), \Lambda^{(i)}(s)) &= \sigma^2 \rho_i\,(s-t) && (2)\end{aligned}$$

[1] A deterministic generalization of binomial cascades was introduced by Coppens and Mandelbrot [CM99]. Their construction of multifractal measures is based on multiplication a periodically extended function with copies itself.

for all t and s in $\mathbb{R}$. In this paper, we assume that the scaled covariance functions ρ_i, $i = 0, 1, \ldots$, satisfy following assumptions:

$$\rho_i(0) = 1, \tag{3}$$
$$\rho_i(x) \geq 0, \quad \forall x \in \mathbb{R}, \tag{4}$$
$$\rho_i(x) = \rho(-x), \quad \forall x \in \mathbb{R}, \tag{5}$$
$$\rho_i(x) \leq \rho_i(y), \quad \text{whenever } |x| > |y|. \tag{6}$$

Thus all the processes are positively correlated and they have the same finite variance σ^2. Furthermore, we assume that correlation decreases as index i grows, more precisely, for all $x \neq 0$

$$\rho_i(x) < \rho_j(x), \qquad \text{whenever } i > j. \tag{7}$$

Such a family can be constructed, for example, from a single positive correlated process with stationary increments simply by changing time scale. In other words, by setting

$$\Lambda^{(i)}(t) \stackrel{d}{=} \Lambda(g(i)t), \quad i = 0, 1, \ldots,$$

where g is an arbitrary positive monotonically increasing unbounded function. Usually we have $g(i) = b^i$, $b > 1$.

Let us consider a product of processes satisfying assumptions (1)–(7)

$$\Lambda_n(t) = \prod_{i=0}^{n} \Lambda^{(i)}(t).$$

This process is almost surely degenerate at the limit, since $\mathbf{E}(\Lambda_n(t)) = 1$ for all n and the limit process $\lim_{n\to\infty} \Lambda_n(t) = 0$ almost surely. The latter is seen by taking logarithm of the product,

$$\log \Lambda_n(t) = \sum_{i=0}^{n} \log \Lambda^{(i)}(t),$$

and interpreting it is as a random walk. Because of the negative drift, this random walk will end up in minus infinity almost surely.

In stead of point wise limits, one can consider the cumulative process

$$A_n(t) = \int_0^t \Lambda_n(s)\ ds = \int_0^t \prod_{i=0}^{n} \Lambda^{(i)}(s)\ ds. \tag{8}$$

and its limit. The main result of this paper is a sufficient condition for existence of a non-degenerate limit A_∞.

Recall the following two basic properties of martingales (for proofs see, e.g. [Wil91]).

Lemma 2.1. *Let M be a supermartingale bounded in $\mathcal{L}_1$: $\sup_n \mathbf{E}(|M_n|) < \infty$. Then, almost surely, $M_\infty = \lim M_n$ exists and is finite.*

Lemma 2.2. *Let $M = (M_n : n \geq 0)$ be a martingale for which $M_n \in \mathcal{L}_2$ for all n. Then M is bounded in $\mathcal{L}_2$ if and only if*

$$\sum \mathbf{E}\left[(M_k - M_{k-1})^2\right] < \infty; \tag{9}$$

and when this obtains $M_n \to M_\infty$ almost surely and in $\mathcal{L}_2$.

Interpreting $A(t) = (A_n(t) : n \geq 0)$ as a discrete martingale, it is straightforward to show that $A_\infty(t)$ exists and is finite almost surely. This does not guarantee the $\mathcal{L}_1$ convergence, as seen in the case of $\Lambda_n(t)$, and we need something more. The question of non-degeneracy in a very general setup is studied by Kahane [Kah87]. He considers random measures $\nu_\infty(E) := \lim_{n\to\infty} \int_E \Lambda_n d\nu$ for arbitrary Radon measures ν and positive martingales Λ_n. In our work, ν is the Lebesgue measure and our interests are in local scaling structure of the limiting process $\int_0^t d\nu_\infty$. To this end, we study the $\mathcal{L}_2$-case only. This is natural since in applications one often relies or is interested in second order statistics.

Proposition 2.1. *For all $t \geq 0$, $A(t) = (A_n(t) : n \geq 0)$ is bounded in $\mathcal{L}_2$ if and only if*

$$\sum_{n=0}^{\infty} a_n(t) < \infty, \tag{10}$$

where $a_n(t) = \int_0^t \rho_n(x) \prod_{i=0}^{n-1}(1+\sigma^2\rho_i(x))dx$. Furthermore, $A_n(t) \to A_\infty(t)$ almost surely and in $\mathcal{L}_2$ if condition (10) holds true.

Proof. Fix $t > 0$. Clearly, by Fubini, $A(t) = \{A_n(t)\}$ is a martingale with respect to $\{(\mathcal{F}_n, \mathbf{P})\}$, where $\mathcal{F}_n = \sigma(\Lambda^{(0)}(t), \Lambda^{(1)}(t), \ldots, \Lambda^{(n)}(t))$ (an increasing sequence of σ-algebras). $A_n(t) \in \mathcal{L}_2$ for all finite n and thus it is enough to study when inequality (9) is valid.

Let us define

$$d_n(t) \doteq \mathbf{E}\left[(A_n(t) - A_{n-1}(t))^2\right].$$

By the definition of A_n and assumptions (1) and (2),

$$\begin{aligned} d_n(t) &= \mathbf{E}\int_0^t\int_0^t \left(\Lambda^{(n)}(t')-1\right)\left(\Lambda^{(n)}(\xi)-1\right)\ \Lambda_{n-1}(t')\ \Lambda_{n-1}(\xi)\ dt'd\xi \\ &= \int_0^t\int_0^t \operatorname{cov}\left(\Lambda^{(n)}(t'), \Lambda^{(n)}(\xi)\right)\ \mathbf{E}\left(\Lambda_{n-1}(t')\Lambda_{n-1}(\xi)\right)\ dt'd\xi \\ &= \int_0^t\int_0^t \sigma^2\rho_n(|t'-\xi|)\prod_{i=0}^{n-1}(1+\sigma^2\rho_i(|t'-\xi|)\ dt'd\xi. \end{aligned}$$

The change of variables, $x = t' - \xi$, gives

$$d_n(t) \;=\; 2\sigma^2 \int_0^t (t-x)\rho_n(x) \prod_{i=0}^{n-1} (1+\sigma^2\rho_i(x))\; dx.$$

Thus,

$$d_n(t) \le 2\sigma^2 t \int_0^t \rho_n(x) \prod_{i=0}^{n-1} (1+\sigma^2\rho_i(x)) dx = 2\sigma^2 t a_n(t)$$

and by assumption (6)

$$d_n(t) \ge \sigma^2 t \int_0^{t/2} \rho_n(x) \prod_{i=0}^{n-1} (1+\sigma^2\rho_i(x)) dx = \sigma^2 t a_n(t/2). \tag{11}$$

Combine the above with $a_n(t) \le 2a_n(t/2)$ to get

$$\frac{1}{2}\sigma^2 t a_n(t) \le d_n(t) \le 2\sigma^2 t a_n(t).$$

Since A_∞ is positive, nondecreasing process, the statement holds for all t. □

In many applications, simplified forms of the previous proposition are more usable. The following lemmas show that $p_i(t) = \int_0^t \rho_i(x)dx$ plays an essential role in the boundedness of $A(t)$.

Corollary 1. *$A(t) = (A_n(t) : n \ge 0)$ is bounded in $\mathcal{L}_2$ if*

$$\sum_{n=0}^{\infty} (1+\sigma^2)^n p_n(t) < \infty. \tag{12}$$

Proof. Since $\rho_i(x) \le 1$ for all x,

$$\begin{aligned} a_n(t) &= \int_0^t \rho_n(x) \prod_{i=0}^{n-1} (1+\sigma^2\rho_i(x))^n\; dx \\ &\le 2\sigma^2 t(1+\sigma^2)^n p_n(t). \end{aligned}$$

□

As a direct application of the previous results consider a family of processes with exponentially decaying covariances.

Corollary 2. *If covariance of process* $\Lambda^{(i)}$ *is*

$$\operatorname{cov}\left(\Lambda^{(i)}(t), \Lambda^{(i)}(s)\right) = \sigma^2 \exp(-\nu b^i |t-s|), \qquad i = 0, 1, \dots ,$$

then $A(t) = (A_n(t))$ *is bounded in* $\mathcal{L}_2$ *if and only if* $b > 1 + \sigma^2$*. When this happens* $A_n \to A_\infty$ *almost surely and in* $\mathcal{L}_2$*.*

Proof. The sufficiency follows directly from corollary 1, just insert $\rho_n(x) = e^{-b^n \nu x}$ into the definition of $p_n(t)$.

To prove necessity we need a little more. Since

$$d_n(t) \;\; \geq \;\; \frac{1}{2}\sigma^2 t \int_0^t e^{-b^n \nu x}(1 + \sigma^2 e^{-\nu b^n x})^n \, \mathrm{d}x,$$

integration gives

$$\begin{aligned} d_n(t) \;\; &\geq \;\; \frac{(1+\sigma^2)}{\nu}\left(1 - \frac{(1+\sigma^2 e^{-\nu b^n t})^{n+1}}{(1+\sigma^2)^{n+1}}\right)\frac{(1+\sigma^2)^n}{(n+1)b^n} \\ &\geq \;\; \frac{1+\sigma^2}{2\nu}\,\frac{(1+\sigma^2)^n}{(n+1)b^n}, \end{aligned} \tag{13}$$

if n is large enough. Thus $\sum d_n(t)$ diverges if $b \leq 1 + \sigma^2$. □

If the covariance decays slower then the convergence condition is naturally stricter.

Corollary 3. *If covariance of process* $\Lambda^{(i)}$ *is*

$$\operatorname{cov}\left(\Lambda^{(i)}(t), \Lambda^{(i)}(s)\right) \sim \sigma^2 |b^i(t-s)|^{-\eta}, \qquad i = 0, 1, \dots , \; (\eta > 0),$$

then $A(t) = (A_n(t))$ *is bounded in* $\mathcal{L}_2$ *if* $b > (1+\sigma^2)^{\max\{1, \frac{1}{\eta}\}}$*. When this happens* $A_n \to A_\infty$ *almost surely and in* $\mathcal{L}_2$*.*

Proof. Two different cases are recognized. If $\eta \in (0,1)$, then the processes are long range dependent, otherwise only sort range dependence is seen. The original scaled covariance function can be bounded above by choosing constant C so that $\tilde{\rho}(x) = \min\{1, C|x|^{-\eta}\} \geq \rho(x)$ for all x. Assume that $t > C^{\frac{1}{\eta}} b^{-n}$, then

$$p_n(t) \leq \int_0^t \tilde{\rho}_n(x)\mathrm{d}x = C^{\frac{1}{\eta}} b^{-n} \frac{\eta}{1-\eta} + C b^{-\eta n} \frac{t^{1-\eta}}{1-\eta}.$$

If $\eta \in (0,1)$ the second term dominates and the $\mathcal{L}_2$ condition is $b > (1+\sigma^2)^{1/\eta}$. Otherwise, the first term is dominant and the condition is same as in the exponential case, i.e., $b > 1 + \sigma^2$. □

The following corollary gives a simple condition when we do not have $\mathcal{L}_2$ convergence.

Corollary 4. *$A = (A_n(t) : n \geq 0)$ is unbounded in $\mathcal{L}_2$ if $\sum_{n=0}^{\infty}(1+\sigma^2\rho_n(t))^n p_n(t)$ diverges.*

Proof. By the decrease of $\rho_i(x)$, $x \in [0, \infty)$,

$$\begin{aligned} a_n(t) &= \int_0^t \rho_n(x) \prod_{i=0}^{n-1}(1+\sigma^2\rho_i(x))^n \, dx \\ &\geq (1+\sigma^2\rho_n(t))^n \int_0^t \rho_n(x)dx. \end{aligned} \tag{14}$$

□

Notice that we have considered only conditions for $\mathcal{L}_2$ boundedness. If $A(t)$ is not bounded in $\mathcal{L}_2$, it is still possible that the limit measure is non-degenerate, i.e., the limit may exist in the $\mathcal{L}_1$ sense. The sufficient condition for $\mathcal{L}_1$ convergence is uniform integrability of the family $\{A_n(t)\}$. The study of this case is left for the future.

3 Product of Markov jump processes

Let us consider product processes which are constructed by taking independent rescaled copies of the original ("mother") process:

$$\Lambda^{(i)}(\cdot) \stackrel{d}{=} \Lambda(b^i\cdot), \quad i = 0, 1, \dots ,$$

where $b > 1$. Assume that the mother process Λ is a stationary piecewise constant Markov process which has independent exponentially distributed constant periods. We allow the length distribution of a constant period to depend on the underlying state of the process, but we require transition rates to be bounded both above and below:

$$\mathbf{P}(\Lambda(t) \text{ constant on } [t, t+\Delta) \mid \Lambda(t) = x) = \exp(-\nu(x)\Delta),$$

where

$$\nu_{\min} \leq \nu(x) \leq \nu_{\max} \qquad \forall\, x. \tag{15}$$

Furthermore, we assume that the variance of Λ is finite and its covariance decays exponentially.

Since the decay of the covariance is exponential, corollary 2 gives a sufficient and necessary condition for L_2 boundedness, i.e., $b > 1 + \sigma^2$.

As the first example, consider a stationary two-state Markov process $\Lambda(t)$ with transition rates ν_1 and ν_2 on the state space $S = \{S_1, S_2\}$. In order to $\mathbf{E}(\Lambda(t)) = 1$ the transitions rates must satisfy the equation

$$\frac{\nu_2 S_1}{\nu_1 + \nu_2} + \frac{\nu_1 S_2}{\nu_1 + \nu_2} = 1.$$

The covariance is given by

$$\mathrm{cov}(\Lambda(t), \Lambda(s)) = \sigma^2 e^{-(\nu_1+\nu_2)|s-t|},$$

where

$$\sigma^2 = \frac{\nu_2 S_1^2 + \nu_1 S_2^2}{\nu_1 + \nu_2} - 1.$$

Constructing a family $\{\Lambda^{(i)}\}$ satisfying (1)-(7) from the "mother processes" Λ by changing time

$$\Lambda^{(i)}(\cdot) \stackrel{d}{=} \Lambda(b^i \cdot), \quad i = 0, 1, \ldots .$$

means that processes $\Lambda^{(i)}$, $i = 0, 1, \ldots$, are independent birth-dead-processes with transition rates $b^i \nu_1$ and $b^i \nu_2$. The cumulative process $A_n(t)$ is as defined in (8). A realization of this construction is seen in figure 1.

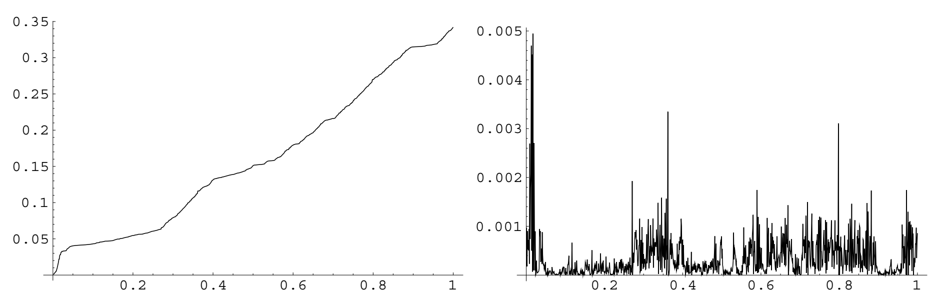

Figure 1: Product of 2-state Markov processes. On the left, a realization of process $A_7(t)$ with $\nu_1 = 2$, $\nu_2 = 1/2$, $S_1 = 1/3$, $S_2 = 7/6$ and $b = 4$. On the right, the corresponding incremental process at resolution 0.001.

As the second example, let the mother process Λ be a piecewise constant process with $\mathrm{Exp}(\nu)$ distributed i.i.d. lengths of periods. For each interval we draw independently a random value M (multiplier) from a common distribution satisfying $\mathbf{E}(M) = 1$. Thus process $\Lambda^{(n)}(t)$ is a piecewise constant process whose covariance is given by

$$\mathrm{cov}(\Lambda^{(n)}(0)\Lambda^{(n)}(x)) = \mathrm{var}(M) e^{-\nu_n |x|} = \sigma^2 \rho_n(x).$$

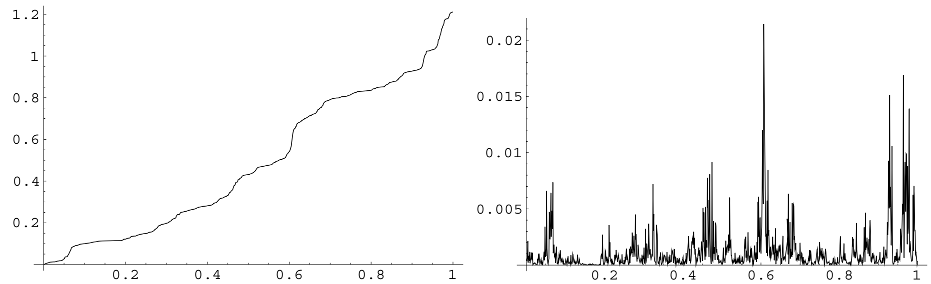

Figure 2: Independent multipliers. On the left, a realization of process $A_7(t)$ with $\nu = 1$, $b = 4$, and multipliers from Gamma(3,1). On the right, the corresponding incremental process at resolution 0.001.

3.1 Some properties

It easy to see that the limit process A_∞ satisfies the "recursive" equation

$$A(t) = \frac{1}{b} \int_0^t \Lambda(s)\ d\tilde{A}(bs), \tag{16}$$

where

(i) the processes Λ and $\tilde{A}$ are independent, and

(ii) the processes A and $\tilde{A}$ are equally distributed.

This is the counterpart to equation (3) in [KP76]. Kahane and Peyriere derive all their results from this equation. As regards their theorem 1, the equivalence of conditions (α), (β) and (γ) is easy to establish in our case as well with exactly same argument as theirs. The main result, the existence he existence condition (δ), remains an open problem in our case. We can only state the existence condition in $\mathcal{L}_2$ sense (see proposition 3.3).

Proposition 3.1. *The following are equivalent:*

(a) the a.s. martingale limit $A_\infty(1)$ *satisfies* $\mathbf{E}A_\infty(1) = 1$*;*

(b) the a.s. martingale limit $A_\infty(1)$ *satisfies* $\mathbf{E}A_\infty(1) > 0$*;*

(c) the equation (16) has a solution such that $\mathbf{E}A(1) = 1$.

Proof. Assume (c). By assumption, $\tilde{A}$ can again be written as

$$\tilde{A}(t) = \frac{1}{b} \int_0^t \tilde{\Lambda}(s)\ d\tilde{\tilde{A}}(bs),$$

where $\tilde{\Lambda}$ is independent from Λ, etc. Denote $\Lambda^{(1)} = \Lambda$, $\Lambda^{(2)} = \tilde{\Lambda}$ etc. Then

$$\mathbf{E}(A(1)|\Lambda^{(1)}, \dots, \Lambda^{(n)}) = \int_0^1 \prod_{i=1}^{n} \Lambda^{(i)}(b^{i-1}s)\ ds.$$

This martingale is uniformly integrable — but it is the same (in distribution at least) as $A_n(1)$! Thus, (a) holds. The remaining implications (a)$\Rightarrow$(b)$\Rightarrow$(c) are obvious. □

Proposition 3.2. *Assume that the non-trivial integrable limit $A = A_\infty$ exists and that $\mathbf{E}\Lambda \log \Lambda < \log b$ and $\mathbf{E}A(1) \log A(1) > \infty$. Then A is continuous.*

Proof. Denote by B the pure jump part of A:

$$B(t) = \sum_{s \leq t} \Delta A(s).$$

By proposition 3.1, (16) holds. Since no "new" jumps can be created in the integration, (16) holds for B as well. Denote $g(x) = x \log x$. Now,

$$\begin{aligned}
\mathbf{E}\sum_{s\leq 1} g(\Delta B(s)) &= \mathbf{E}\sum_{s\leq 1} g\left(\frac{1}{b}\Lambda(s)\Delta\tilde{B}(bs)\right) \\
&= b\mathbf{E}\sum_{s\leq 1} g\left(\frac{1}{b}\Lambda(s/b)\Delta\tilde{B}(s)\right) \\
&= b\mathbf{E}\sum_{s\leq 1}\left[\frac{1}{b}\Lambda(s/b)\Delta B(s)(\log\Lambda(s/b) + \log\frac{\Delta B(s)}{b})\right] \\
&= \mathbf{E}\sum_{s\leq 1}\Lambda(s/b)\Delta B(s)\log\frac{\Delta B(s)}{b} + \mathbf{E}\sum_{s\leq 1}(\Lambda(s/b)\log\Lambda(s/b))\Delta B(s) \\
&= \mathbf{E}\sum_{s\leq 1} g(\Delta B(s)) - \log(b)\mathbf{E}B(1) + \mathbf{E}(\Lambda\log\Lambda)\mathbf{E}B(1).
\end{aligned}$$

where the second equality uses the fact that B has stationary increments. We have $\mathbf{E}B(1) \leq \mathbf{E}A(1) < \infty$ and, since g is superadditive, $\mathbf{E}\sum_{s\leq 1} g(\Delta B(s)) \leq \mathbf{E}g(B(1)) \leq \mathbf{E}g(A(1)) < \infty$. If B is non-trivial, it follows that $\mathbf{E}\Lambda\log\Lambda = \log b$, which is against assumptions. Thus, $B \equiv 0$. □

Proposition 3.3. *$\{A_n(t)\}$ is bounded in $\mathcal{L}_2$ if and only if $b > 1 + \sigma^2$. When this happens $A_n \to A_\infty$ almost surely and in $\mathcal{L}_2$.*

Proof. See corollary 2. □

In order to be able to say more about the behavior of the second moment, we need the following technical lemma.

Lemma 3.1. *If* $\mathrm{cov}(\Lambda(s),\Lambda(t)) = \sigma^2 \exp(-\nu|t-s|)$ *for some constants* $\sigma^2 < \infty$, $0 < \nu < \infty$, *then*

$$\mathbf{E}\left[\left(A_\infty(t) - \frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt)\right)^2\right] \le Ct\mathbf{E}\left[A_\infty(bt)^2\right], \tag{17}$$

where

$$\tilde{A}_\infty(t) = \lim_{n\to\infty}\int_0^t \prod_{i=1}^n \Lambda^{(i)}(x/b)\mathrm{d}x.$$

Proof. By the recursion equation (16)

$$A_\infty(t) - \frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt) \;=\; \frac{1}{b}\int_0^t (\Lambda^{(0)}(s) - \Lambda^{(0)}(0))\; d\tilde{A}_\infty(bs).$$

Denote $Z(s) = \Lambda^{(0)}(s) - \Lambda^{(0)}(0)$. If $s_1, s_2 \le t$, then

$$\begin{aligned}\mathbf{E}Z(s_1)Z(s_2) &= \sigma^2(\rho(0) + \rho(s_1 - s_2) - \rho(s_1) - \rho(s_2)) \\ &\le 2\sigma^2(1-\rho(t)) \le \sigma^2\nu t\end{aligned}$$

Thus

$$\begin{aligned}&\mathbf{E}\left[(A_\infty(t) - \frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt))^2\right] \\ &= \frac{1}{b^2}\mathbf{E}\left[\int_0^t\int_0^t (Z(s_1)Z(s_2))\; d\tilde{A}_\infty(bs_1)d\tilde{A}_\infty(bs_2)\right] \\ &= \frac{1}{b^2}\mathbf{E}\mathbf{E}\left[\int_0^t\int_0^t (Z(s_1)Z(s_2))\; d\tilde{A}_\infty(bs_1)d\tilde{A}_\infty(bs_2) \mid \sigma(\tilde{A}_\infty)\right] \\ &= \frac{1}{b^2}\mathbf{E}\left[\int_0^t\int_0^t \mathbf{E}\left((Z(s_1)Z(s_2))\;\; d\tilde{A}_\infty(bs_1)d\tilde{A}_\infty(bs_2)\right]\right. \\ &\le \frac{\sigma^2\nu t}{b^2}\mathbf{E}\left[\tilde{A}_\infty(bt)^2\right].\end{aligned}$$

□

Proposition 3.4. *There exist constants* C_1 *and* C_2 *such that*

$$C_1 t^{1+\delta} \le EA_\infty(t)^2 \le C_2 t^{1+\delta}, \quad t\in[0,1],$$

where $\delta = 1 - \log_b \mathbf{E}\Lambda^2$.

Proof. Let us write the process $A_\infty(t)$ in two parts

$$A_\infty(t) = \frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt) + \left(A_\infty(t) - \frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt)\right),$$

where

$$\tilde{A}_\infty(t) = \lim_{n\to\infty}\int_0^t \prod_{i=1}^n \Lambda^{(i)}(x/b)\mathrm{d}x.$$

Denote $g(t) = \mathbf{E}A_\infty^2(t)$. Then

$$\begin{aligned} g(t) &= \frac{1+\sigma^2}{b^2}\mathbf{E}A_\infty^2(bt) + h(t) \\ &= \frac{1+\sigma^2}{b^2}g(bt) + h(t), \end{aligned}$$

where

$$\begin{aligned} h(t) &= \mathbf{E}\left(A_\infty(t) - \frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt)\right)^2 \\ &\quad +2\mathbf{E}\left[\frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt)\left(A_\infty(t) - \frac{\Lambda^{(0)}(0)}{b}\tilde{A}_\infty(bt)\right)\right] \end{aligned}$$

Let us first study how $g(t)$ behaves when $t = b^{-n}$. Consider the ratio

$$\frac{g(t)}{g(bt)} = \frac{1+\sigma^2}{b^2} + \frac{h(t)}{g(bt)},$$

denote $x_n = g(b^{-n})$ and $z_n = \frac{h(b^{-n})}{g(b^{-n+1})}$, and take logarithm:

$$\begin{aligned} \log x_n &= \log x_{n-1} + \log\left(\frac{1+\sigma^2}{b^2} + z_n\right) \\ &= \log x_0 + \sum_{i=1}^n \log\left(\frac{1+\sigma^2}{b^2} + z_i\right) \\ &= \log x_0 + n\log\frac{1+\sigma^2}{b^2} + \sum_{i=1}^n \log\left(1 + \frac{b^2}{1+\sigma^2}z_n\right) \\ &= b_n + n\log\frac{1+\sigma^2}{b^2}, \end{aligned}$$

where

$$b_n = \log x_0 + \sum_{i=1}^n \log\left(1 + \frac{b^2}{1+\sigma^2}z_n\right).$$

Thus

$$g(b^{-n}) = e^{b_n} \left(\frac{1+\sigma^2}{b^2} \right)^n .$$

Trivially, $e^{b_n} \geq x_0 = \mathbf{E} A_\infty(1)^2$. Neither the upper bound is too difficult to find. By Hölder's inequality and lemma 3.1, $h(t) \leq C g(bt)(t + \sqrt{t})$, thus $z_n \leq c(b^{-n} + b^{-n/2})$ and

$$\begin{aligned} b_n &= \log x_0 + \sum_{i=1}^{n} \log \left(1 + \frac{b^2}{1+\sigma^2} z_n \right) \\ &\leq \log x_0 + \int_1^\infty \log(1 + C b^{-x/2}) \mathrm{d}x \\ &\leq \log x_0 + \int_1^\infty C e^{-\frac{1}{2} x \log b} \mathrm{d}x \\ &\leq \log x_0 + \frac{2C}{\log b}. \end{aligned}$$

Next transform back by writing $b^{-n} = t_n$:

$$g(t_n) = e^{b_n(t_n)} \left(\frac{1+\sigma^2}{b^2} \right)^{-\log t_n / \log b} = B(t_n) t^{2 - \frac{\log(1+\sigma^2)}{\log b}} = B(t_n) t_n^{1+\delta},$$

where $B(t)$ is bounded and $\delta = 1 - \frac{\log(1+\sigma^2)}{\log b}$. Then it is an easy exercise show that the claim holds for an arbitrary $t \in [0,1]$. □

The processes $\Lambda^{(i)}$, $i = 0, 1, \ldots n-1$ determine the *natural partitioning* of the real axis $J^n = \{J_i^n\}$, where J_k^n's are the largest intervals with $\Lambda_n(t) = \prod_{i=0}^{n-1} \Lambda^{(i)}(t)$ constant. We index intervals from left to right and J_1^n denotes the interval that contains the origin. A suitable scaling gives us a strong control over the lengths of intervals on different partitions.

Lemma 3.2. *For all n, there exists i.i.d.* $\mathrm{Exp}(\nu_{\max}/(b-1))$ *distributed random variables $\underline{X}_k$ and i.i.d.* $\mathrm{Exp}(\nu_{\min}/b)$ *distributed random variables $\overline{X}_k$ such that*

$$\underline{X}_k \leq b^n |J_k^n| \leq \overline{X}_k,$$

for all k.

Proof. Let n be fixed. Given the states of processes $\Lambda^{(i)}$, $i = 0, \ldots, n-1$, on the intervals J_k^n, the interval lengths $|J_k^n|$ are independent exponentially distributed random variables. Let $Z_{n,k}$ denote states of the n first processes on interval J_k^n, i.e.,

$$Z_{n,k} = (\Lambda^{(0)}(J_k^n), \ldots, \Lambda^{(n-1)}(J_k^n)).$$

The transition rate from state $Z_{n,k}$ is denoted by $\nu(Z_{n,k})$. Then the cumulative distribution function (cdf) of $b^n|J_k^n|$ is

$$F_{Z_{n,k}}(x) = \mathbf{P}(b^n|J_k^n| < x) = 1 - \exp(-\nu(Z_{n,k})x).$$

Although ν depends on the underlying state process, it is always bounded above

$$\nu(Z_{n,k}) \leq \sum_{i=0}^{n-1} b^i \nu_{\max} = \frac{1-b^n}{1-b}\nu_{\max} \leq \nu_{\max}/(b-1)$$

(all processes in the most rapidly changing state). Correspondingly, ν is bounded below

$$\nu(Z_{n,k}) \geq \sum_{i=0}^{n-1} b^i \nu_{\min} = \frac{1-b^n}{1-b}\nu_{\min} \geq \nu_{\min}/b$$

(all processes in the most stable state).

When generating process $\{|J_k^n|\}$, we have freedom to choose whether to first draw the length of an interval and after that the state on the next interval, or vice versa. For our purposes the former suits better. With the knowledge of the states of $\Lambda^{(i)}$'s on J_k^n , $|J_k^n|$ can be generated from Uniform(0,1) distributed random variable $\{U_k\}$ by setting $b^n|J_k^n| = F_{Z_{n,k}}^{-1}(U_k)$. Associate a sequence of independent random variables $\{U_k\}$ with sequences $\{\underline{X}_k\}$ and $\{\overline{X}_k\}$ by setting $\underline{X}_k = G^{-1}(U_k)$ and $\overline{X}_k = F^{-1}(U_k)$, where

$$\begin{cases} G(x) & = 1 - \exp\left(-\frac{\nu_{max}x}{b-1}\right) \\ F(x) & = 1 - \exp\left(-\frac{\nu_{min}x}{b}\right). \end{cases}$$

By the very definition $\overline{X}_k$'s are mutually independent, and so are $\underline{X}_k$'s too. From

$$F(x) \leq F_{Z_{n,k}}(x) \leq G(x) \quad \text{for all } \; x \geq 0,$$

follows that

$$\underline{X}_k \leq b^n|J_k^n| \leq \overline{X}_k$$

for all k. □

The above lemma means that we can approximate each partitioning pathwise by a Poisson process with intensity $\approx b^n$.

4 Multifractal analysis

This section is strongly under construction. At the present stage, only some basic ideas are shown and the details will be found in [MNR99]. The goal is to developed multifractal machinery to deal with processes which automatically form natural partitionings. The preliminary results given here concern only processes constructed from a product of pathwise constant Markov jump processes.

Let A_∞ be generated by a product of piecewise constant processes and consider the restriction of A_∞ on the interval $[0, 1]$. Each realization determines a random measure

$$\mu_\infty([0, t)) \doteq A_\infty(t).$$

An essential property of our processes is their multifractal scaling structure. As we already alluded to, the process A_∞ is nowhere differentiable, in other words $\Lambda_n(t)$ converges either to 0 or ∞. With this in mind it is most informative to study the local regularity of A_∞. To this end, we set

$$\alpha_n(t) := \frac{\log \mu_\infty\left(J_n(t)\right)}{\log |J_n(t)|}, \tag{18}$$

where $J_n(t)$ is the unique interval J_k^n containing t. Then, the local Hölder exponent of μ_∞ at t is defined as

$$\alpha(t) := \lim_{n \to \infty} \alpha_n(t). \tag{19}$$

One should think of $\alpha(t)$ as giving approximately[2] the degree of Hölder regularity of A_∞ at t.

A process is said to be multifractal, if $\alpha(t)$ changes erratically in time, more precisely, the sets

$$K_\alpha := \{t \,:\, \alpha_n(t) \to \alpha\} \tag{20}$$

are all dense and, thus, highly interwoven. Assuming stationarity of increments only one of them can have full Lebesgue measure and the others must, hence, have dimension strictly less than one. The *multifractal spectrum*

$$\dim(K_\alpha) \tag{21}$$

as a function of α gives a compact description of the 'size' of K_α and the multifractal structure of μ_∞.

[2]Here we approximate balls with random intervals $J^n(t)$. We are working on proofing that $\alpha(t)$ gives exact Hölder regularity expect for a set of dimension zero if lengths of intervals J_k^n behave enough uniformly.

To obtain a formula on $\dim(K_\alpha)$ we imitate the usual procedure of multifractal analysis which is to take q-th powers of the multipliers $\Lambda^{(i)}$ and to consider the so obtained limiting measure $\overline{\mu}$. In order to have convergence we define the auxiliary function

$$\beta(q) := q - 1 - \log_b \mathbb{E}[\Lambda^q] \tag{22}$$

and let $\overline{\Lambda} := \Lambda^q b^{\beta(q)-q+1}$. The peculiar form of β and $\overline{\Lambda}$ will become clear instantly.

The basic tool for obtaining $\dim(K_\alpha)$ is a Frostman type lemma: One shows that for $\alpha = \beta'(q)$ the set K_α has (almost surely) full $\overline{\mu}$-mass and that the local scaling exponent of $\overline{\mu}$ is[3] $\beta^*(\alpha)$. The usual Frostman lemma allows then to conclude that the Hausdorff dimension of K_α is bounded below by $\beta^*(\alpha)$. This explains the peculiar form of β.

However, care has to be taken in this result as it speaks actually about the Hausdorff dimension of K_α which is computed by the random partition J_k^n. In order to obtain the usual Hausdorff dimension we define

$$\tilde{K}_\alpha := \{t \,:\, \alpha_n(t) \to \alpha \text{ and } (1/n)\log|J_n(t)| \to -\log(b)\} \tag{23}$$

The global scaling properties are captured by the asymptotics of ensemble moments. We can deal either with the expectations

$$T(q) := \inf\{\gamma : \mathbf{E} \sum_{J_k^n \subset [0,1]} \mu_\infty(J_k^n)^q |J_k^n|^{-\gamma} \to 0 \text{ as } n \to \infty\},$$

or pathwise notions

$$\tau(q) := \inf\{\gamma : \sum_{J_k^n \subset [0,1]} \mu_\infty(J_k^n)^q |J_k^n|^{-\gamma} \to 0 \text{ as } n \to \infty\}.$$

Usually the following inequalities hold

$$T^*(\alpha) \geq \tau^*(\alpha) \geq \dim(K_\alpha) \geq \dim(\tilde{K}_\alpha) \geq \beta^*(\alpha).$$

We conjecture that in the case of piecewise constant Markov processes, one can show that, instead of inequalities, all the above multifractal spectra are equal. Unfortunately, some details ares still missing in the proof.

5 Concluding remarks

The mathematical analysis of multifractal products of stochastic processes is far from complete. [MNR99] tries to give a rigorous treatment to some

[3] Here β^* is the Legendre transform of β: $\beta^*(\alpha) = \inf_q(q\alpha - \beta(q))$.

easy cases like piecewise continuous processes with exponentially distributed sojourn times. Extending to long range dependent processes and maybe even to continuous processes is worth of own study. Another interesting question is the distribution of A_∞: it would be quite natural that in many cases it would be approximately log-normal.

The second path leads to engineering-applications. Modeling of real traffic based on multifractal products of stochastic processes is a topic which seems to be very promising. As a matter of fact, there have been some plans to look at things like queuing experiments, parameter estimators and synthesis algorithms for matching real data traffic.

References

[AP96] M. Arbeiter and N. Patzchke. Random self-similar multifractals. *Math. Nachr.*, 181:5–42, 1996.

[CM99] M.-O. Coppens and B. Mandelbrot. Easy and natural generation of multifractals: multiplying harmonics of periodic functions. In M. Dekking, J. Lévy Véhel, and C. Tricot, editors, *Fractals: theory and applications in engineering*. Springer, 1999.

[Fal94] K. Falconer. The multifractal spectrum of statistically self-similar measures. *J. Theor. Prob.*, 7:681–702, 1994.

[FGW98] A. Feldmann, A. C. Gilbert, and W. Willinger. Datanetworks as cascades: Investigating the multifractal nature of Internet WAN traffic. In *Proceedings of the ACM/SIGCOMM'98*, Vancouver, Canada, 1998.

[Jaf96] S. Jaffard. Sur la nature multifractale des processus du Lévy. *C. R. Acad. Sci. Paris*, 323:1059–1064, 1996.

[Kah87] J.-P. Kahane. Positive martingales and random measures. *Chin. Ann. of Math.*, 8B:1–12, 1987.

[KP76] J.-P. Kahane and J. Peyriére. Sur certaines martingales de Benoit Mandelbrot. *Advances in mathematics*, 22:131–145, 1976.

[LVR97] J. Lévy Véhel and R. Riedi. Fractional brownian motion and data traffic modeling: The other end of the spectrum. In *Fractals in Engineering 97*, pages 185–202. Springer, 1997.

[Man72] B. B. Mandelbrot. Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence. In M. Rosenblatt and C. Van Atta, editors, *Statistical models and turbulence*, number 12 in Lecture notes in physics, pages 331–351. Springer, 1972.

[Man74] B. B. Mandelbrot. Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. *J. Fluid. Mech.*, 64, 1974.

[MN97] P. Mannersalo and I. Norros. Multifractal analysis of real ATM traffic: a first look. COST257TD(97)19, VTT Information Technology, 1997.

[MNR99] P. Mannersalo, I. Norros, and R. Riedi. Multifractal products of stochastic processes. In preparation, 1999.

[Pat97] N. Patzchke. Self-conformal multifractal measures. *Advances in Applied Mathematics*, 19:486–513, 1997.

[RCRB98] R. Riedi, M. Crouse, V. Ribeiro, and R. Baraniuk. A multifractal wavelet model with application to TCP network traffic. *IEEE Special Issue on Information Theory*, 1998. To appear.

[RLV97] R. Riedi and J. Lévy Véhel. TCP traffic is multifractal: a numerical study. Inria research report, no. 3129, Project Fractales, INRIA Rocquencourt, 1997.

[RW94] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes and martingales*, volume Volume 1: Foundations. Wiley, 2nd edition, 1994.

[Wil91] D. Williams. *Probability with Martingales.* Cambridge University Press, 1991.