Capturing Market Dynamics

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Market models

The ongoing discussion about the role of LRD, fBm and cascades in Financial data
History

- **Bachelier 1900**
  - Computes “Barrier Option” for price modeled by Brownian motion
- **Mandelbrot 1962:**
  - self-similarity of Cotton prices
- **Samuelson**
  - *Foundations of Economic Analysis* (1947)
  - keep only minimum of simple economic relations;
  - rewrite it as a constrained optimization problem.
- **Black-Scholes(’73)-Merton(’70s) (Nobel 1997)**
  - Log returns are BM (returns are geometric Brownian motion)
  - Derive partial differential equation for option price
- **Girsanov (’60) theorem**
  - Equivalent martingale measure (pricing measure) simplifies the computation of option prices
- **Engle (Nobel 2003)**
  - ARCH, Cond.Duration and other time series models
Ongoing

• Estimating Fractal Dimension Of S&P 500 Using Wavelet Analysis
  • Bayraktar, Poor, Sircar 2003

• Dynamics of financial markets – Mandelbrot’s cascades and beyond.
  • Muzy et al 2005

• London Stock Exchange: signs of orders obey LRD with $H = 0.7$.
  – This would suggest a very strong market inefficiency.
  – However, compensated for by anti-correlated fluctuations in transaction size and liquidity
    • Lillo, Farmer 2004

• Why Stock Markets Crash
  – Local logarithmic chirps
    • Sornette
fBm and arbitrage

• Arbitrage strategy for Black-Scholes model driven by
  – fractional Brownian motion or
  – by a time changed fractional Brownian motion,
  – when the volatility is stochastic.
    • Bayraktar, Poor 2005
  – fBm is not semi-martingale, nor a Markov process
  – Black-Scholes differential equation not well-defined
  – No equivalent martingale measure for fBm, and therefore in a frictionless market where continuous trading is possible there exist arbitrage strategies.
fBm and Stochastic Calculus

- Girsanov formula, Ito formula for fBm
  - Conditional expectation w.r.t. fBm
    - Gripenberg and Norros.
    - Says that you need 1 minute of past to predict a minute of future
  - On Girsanov and equivalent martingale
    - Norros et al 2005
    - Decreusefond and Ustunel 2005

Illustrating Girsanov’s equivalent probability measure: Left: BM with drift under natural probability. Right: paths colored according to Girsanov’s equiv. probab. Measure [under which the trend is removed]
Conclusion: we need

• Estimation procedures
  For LRD and multifractal scaling
• new improved models
Multifractal estimation

Large Deviations
Multifractal formalism
Legendre transform
Counting via Large Deviations

- **Notation:**
  - Number of dyadic intervals with exponent $\sim a$:
    \[ N_{n,\delta}(a) := \# \{(\epsilon_1 \ldots \epsilon_n) : a-\delta \leq \alpha_n(\epsilon_1 \ldots \epsilon_n) < a+\delta\} \]
  - Partition sum: a microscope inspired by LDP
    \[ S_n(q) := \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q = \sum_{\epsilon_1 \ldots \epsilon_n} |2^n| q \alpha_n(\epsilon_1 \ldots \epsilon_n) \]
  - Assume powerlaws:
    \[ N_{n,\delta}(a) \sim 2^{nf(a)} \quad S_n(q) \sim 2^{-n\tau(q)} \]
  - Typically (LDP)
    \[ f(a) = \inf_q (qa - \tau(q)) \]
Multifractal estimation via wavelets

- Set indicator of oscillation to be wavelet coefficient:

\[ I(\epsilon_1 \ldots \epsilon_n) := [t_n, t_n + 1/2^n) \] \quad t_n := \sum_{k=1}^{n} \epsilon_k/2^k

\[ \Delta I_n(\epsilon_1 \ldots \epsilon_n) = d_{2^n t_n, -n} \]

- Log-log regression of \( S \) yields estimate

\[ S_n(q) := \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q = \sum_j |d_j, -n|^q \sim 2^{-n\tau(q)} \]

- From before: \( \tau(2) \) estimates \( 2H-1 \) for fBm
  - ...as it should since \( \tau(q) = qH-1 \)
fBm simulation
Fractional ARIMA

• Recall the definition which allows for sequential simulation

\[ X_i = \Delta^{-d} \epsilon_i, \quad i \geq 1, \]

• ...and has simpler spectral density than fGn

\[
f(\lambda) = \frac{1}{2\pi} \left(2 \sin \frac{\lambda}{2}\right)^{-2d} \sim \frac{1}{2\pi} |\lambda|^{-2d} \text{ as } \lambda \to 0.
\]
Fast exact Synthesis of fBm

• To produce exact correlation structure need to decompose auto-correlation matrix

• “Naïve” Decomposition requires $O(N^3)$
  – Allows only sample size of a few hundred

• Cholesky: $O(N^2)$
  – Restrict to synthesis length of $\sim 1000$

• Circulant method:
  – Embed correlation matrix in circulant matrix
  – Decomposition is then exactly the FFT
  – Allows to synthesize fGn of $\sim 1'000'000$

• Large simulated traces needed for “trading time models” (see next)
Multifractal Subordination

Processes with multifractal oscillations
Multifractal time warp

\( B_H(M(t)) \): \( B_H \) fBm, \( dM \) independent measure

A versatile model

- \( M(t) \): Multifractal
  Time change
  Trading time

- \( B \): Brownian motion
  Gaussian fluctuations
Hölder regularity

- **Levy modulus of continuity:**
  - With probability one for all $t$
    \[ |B_H(t + \delta) - B_H(t)| \sim |\delta|^H \]
  - Thus, exponent gets stretched:
    \[ |B_H(M(t+\delta)) - B_H(M(t))| \sim |M(t+\delta) - M(t)|^H \sim |\delta|^{H\alpha(t)} \]
  - and spectrum gets squeezed:
    \[ \dim E_a[B_H(M)] = \dim E_{a/H}[M] \]
Auto-Correlation of $B(M(t))$

- Conditional on $M$: from auto-correlation of $B$
  \[
  \mathbb{E}[B(M(t))B(M(s))|M] = (\sigma^2/2)[M^{2H}(t) + M^{2H}(s) - |M(t) - M(s)|^{2H}]
  \]

- For the increments $X(t) + B(M(t+1)) - B(M(t))$
  \[
  \mathbb{E}[X(t)X(s)|M] = (\sigma^2/2) \left( [M(t+1) - M(s)]^{2H} - [M(t) - M(s)]^{2H} \right.
  - [M(t+1) - M(s+1)]^{2H} + [M(t) - M(s+1)]^{2H} \left. \right)
  \]

- Assume $H = 1/2$ and $\mathbb{E}[(M(t))] = t$. Then
  \[
  \mathbb{E}[B(M(t))B(M(s))] = \sigma^2 \min(\mathbb{E}[(M(s))], \mathbb{E}[(M(t))]) = \sigma^2 \min(s, t)
  \]
  \[
  \mathbb{E}[X(t+k)X(t)] = \mathbb{E}[M(k+1) - 2M(k) + M(k-1)] = 0
  \]

- Uncorrelated, 2nd order stationary, but not Gaussian
- Dependence of higher order through $M(t)$
- Blind spot of spectral analysis
Estimation: Wavelets decorrelate

(with P. Goncalves)

Decorrelation:

\[ \mathbb{E}[W_{jk} W_{jm}] \]

\[ \overset{1}{=} \mathbb{E} \int \int \psi_{jk}(t) \psi_{jm}(s) B(M(t)) B(M(s)) dt ds \]

\[ \overset{2}{=} \left( \sigma^2 / 2 \right) \int \int \psi_{jk}(t) \psi_{jm}(s) \mathbb{E} \left[ M^{2H}(t) + M^{2H}(s) - |M(t) - M(s)|^{2H} \right] dt ds \]

\[ \overset{3}{=} -\left( \sigma^2 / 2 \right) \int \int \psi_{jk}(t) \psi_{jm}(s) \mathbb{E} \left[ |M(t - s)|^{2H} \right] dt ds \]

\[ \overset{4}{=} -\left( \sigma^2 / 2 \right) \int |2^j \tau|^{T(2H)} \gamma_{\psi}(\tau - (k - m)) d\tau \]

\[ \overset{5}{=} -\left( \sigma^2 / 2 \right) |2^j|^{T(2H)} \int \left[ \frac{ |\Psi(\nu)|^2 }{|\nu|^{T(2H)+1}} \right] e^{-i2\pi\nu(k-m)} d\nu, \]

\[ \overset{6}{\sim} O \left( |k-m|^{T(2H)+1-2N} \right) \quad (|k-m| \to \infty) \]

(1) \( W_{jk} = \int \psi_{jk}(t) B(M(t)) dt \)

(2) \( \mathbb{E}[B(u) B(v)] = (\sigma^2 / 2) [u^{2H} + v^{2H} - |u-v|^{2H}] \)

(3) \( \int \psi(s) ds = 0 = \int \psi(t) dt, \)

(4) \( \psi_{jk}(t) = 2^j \int \psi(t/2^j - k), \quad \tau = 2^{-j}(t-s), \)

\( \gamma_{\psi}(\tau) = \int \psi(i) \psi(i+\tau) di, \quad \mathbb{E}[M(h)] = h^{T(q)} \)

(5) Parceval

(6) \( \Psi(\nu) \sim |\nu|^N \)
Multifractal Estimation for $B(M(t))$

- Weak Correlations of Wavelet-Coefficients:
  (with P. Goncalves)

- Improved estimator due to weak correlations

- Multifractal Spectrum

\[ M(t+s) - M(t) \sim s^{a(t)} \]
\[ B(t+u) - B(t) \sim u^H \quad (\forall \ t) \]
\[ \rightarrow \]
\[ B(M(t+s)) - B(M(t)) \sim s^{H*a(t)} \]

Estimation
Multifractal time in Zoology

- From Random Walk to Multifractal Random Walk in Zooplankton Swimming Behavior
  - Seuront, Schmitt, Brewer, Strickler and Sami 2004

Fig. 1. Examples of distance traveled by *Temora longicornis* (12.5-Hz resolution, A) and *Daphnia pulex* (10-Hz resolution, B), compared to the distance traveled in the case of pure random motion (C).
Tail estimation

And diverging moments
Diverging Moments

Diverging moments: \[ \mathbb{E}|X|^q = \infty \]

They bear on...

• Estimation of tails: \[ P[|X| > x] \sim x^{-\alpha} \]

• Estimators per se:
  - Bias \[ (X_1^2 + \ldots X_n^2)/n \]
  - Asymptotic normality
Theory
• Let $\lambda > 0$.

$$\mathbb{E}[|X|^r] < \infty \text{ for all } r < \lambda$$

$\iff$

$$P[|X| > 1/u] \overset{u \to 0}{\to} O(|u|^r) \text{ for all } r < \lambda$$

• “Finiteness of Moments and Tails go together”
Characteristic function 101

• Characteristic function:

\[ \phi(u) = \mathbb{E}[\exp(iuX)] \]

• Char Fct and Moments 101: If \( E|X|^n \) is exists then

\[ \phi^{(n)}(0) = i^n \mathbb{E}[X^n] \]

• Vice versa: If \( \phi \) has 2p derivates then \( E|X|^{2p} \) exists
Characteristic function 102

- Char Fct and Moments 102: (Tauberian Theorem)
  For $0 < \lambda < 2$
  \[ \mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda \]
  \[ \Re \phi(u) - 1 \xrightarrow{u \to 0} O(|u|^r) \quad \text{for all } r < \lambda \]

- Example: symmetric stable laws (moments up to $\alpha$)
  \[ \phi(u) = \Re \phi(u) = \exp(-|u|^{\alpha}) \approx 1 - |u|^\alpha + \ldots \quad (|u| \to 0) \]

\[ \alpha = 0.8 \]
\[ \alpha = 1: \text{ Cauchy} \]
\[ \alpha = 1.8 \]
Extension to orders $> 2$

- Kawata ('72) / Lukacs ('83) / Ramachandran ('69):
  - Let $2p < \lambda \leq 2p + 2$ with integer $p$.

\[
\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda
\]

\[
\mathbb{E}[|X|^{2p}] < \infty, \text{ i.e. } \phi^{2p}(0) \text{ exists, and}
\]

\[
\text{Re } \phi(u) - \sum_{k=1}^{p} \frac{(-1)^k}{(2k)!} \mathbb{E}[X^{2k}] u^{2k} \overset{u \to 0}{\to} O(|u|^r) \quad \text{for all } r < \lambda
\]
Estimating the Regularity of $\phi$

- **Motivation:**
  - exact regularity of $\phi$ at zero provides the cutoff value for finite moments

- **Microscope for regularity:** Wavelet transform $T$

\[
T(a, b) = \langle \text{Re } \phi, \psi_{a,b} \rangle = \int \text{Re } \phi(s) \cdot \frac{1}{a} \psi \left( \frac{s-b}{a} \right) ds
\]

- **Simplified regularity theorem:** Assume
  - Wavelet regularity $N > \lambda$
  - Hölder polynomial $P_\phi$ of $c \int t^k \psi(t) dt = 0$ for $0 \leq k < N$
  - Transform $T(a, b)$ is maximal at $b=0$
  - Then

\[
\text{Re } \phi(u) - P_\phi(u) \underset{u \to 0}{\rightarrow} O(|u|^r) \quad \text{for all } r < \lambda
\]
\[
\Leftrightarrow \quad T(a, 0) \underset{a \to 0}{\rightarrow} O(|a|^r) \quad \text{for all } r < \lambda
\]
Simplified Proof

• If

\[
\phi(u) - P_\phi(u) \xrightarrow{u \to 0} O(|u|^r)
\]

and if

(2) the wavelet \( \psi \) is supported on \([0,1]\)

• then

\[
T(a, 0) \xrightarrow{a \to 0} O(|a|^r)
\]

\[
|T(a, 0)| = \left| \langle \phi, \psi_{a,0} \rangle \right| \equiv \left| \frac{1}{a} \int_0^a \phi(s) \psi(s/a) \, ds \right|
\]

\[
(2) \quad = \left| \frac{1}{a} \int_0^a (\phi(s) - P_\phi(s)) \psi(s/a) \, ds \right|
\]

\[
\int t^k \psi(t) \, dt = 0 \quad \text{for } 0 \leq k < N
\]

\[
(1) \quad \leq C \cdot \frac{1}{a} \int_0^a |s|^r |\psi(s/a)| \, ds
\]

\[
\leq C \cdot a^r \frac{1}{a} \int_0^a |\psi(s/a)| \, ds
\]

\[
\leq C \cdot a^r \cdot \int_{\mathbb{R}} |\psi(s)| \, ds
\]
Wavelet Transform of $\phi$

- Fourier transform:
  \[
  (i) \quad \Psi_{a,b}(x) = \frac{1}{a} \int \psi \left( \frac{s-b}{a} \right) e^{ixs} \, ds = \int \psi(u) e^{i(au+b)x} \, du = e^{ixb} \Psi(ax)
  \]
- Parseval:
  \[
  (ii) \quad T(a,b) = \langle \mathcal{R}e \phi, \psi_{a,b} \rangle = \mathcal{R}e \langle F, \Psi_{a,b} \rangle = \mathcal{R}e \mathbb{E}[\Psi_{a,b}(X)]
  \]

- Assume: Fourier Transform $\Psi$ of $\psi$ is real positive.
  - then:
    \[
    |T(a,b)| \leq \mathbb{E}[|\Psi_{a,b}(X)|] = \mathbb{E}[|\Psi(aX)|] = |T(a,0)|
    \]
  - in other words: $|T(a,b)|$ maximal at $b=0$

- Ex:
  \[
  \Psi(u) = u^{2n} \exp(-u^2) \geq 0 \\
  \psi(x) = (-1)^n \left( \frac{d}{dx} \right)^{2n} \exp(-x^2)
  \]
Wavelet Transform of $\phi$

- **Summary:** Assume: $\Psi$ is real positive
  - Then $|T(a,b)|$ maximal at $b=0$
  - ... and

$$ (iii) \quad T(a,0) = \Re \mathbb{E}[^{\Psi}_{a,0}(X)] = \Re \mathbb{E}[\psi(ax)] = \mathbb{E}[\psi(ax)] $$

- Recall equivalent conditions for $0<\lambda<2$:

  $$(1) \quad \Re \phi(u) - 1 \xrightarrow{u \to 0} O(|u|^r) \quad \text{for all } r < \lambda $$

  $$(2) \quad T(a,0) \xrightarrow{a \to 0} O(|a|^r) \quad \text{for all } r < \lambda $$

- $\rightarrow$ estimate regularity of $\Re(\phi)$ by the powerlaw

$$ |T(a,0)| = \mathbb{E}[\psi(aX)] \sim a^\lambda $$
Implementation and Performance
Numerical Implementation

The estimator of $T(a,0)$ of $\phi$ is

- ...simple:

$$T(a,0) = \mathbb{E}[\psi(aX)] = a^\lambda$$

- ...unbiased

- ...non-parametric!

- Estimation of critical order $\lambda = \sup\{q: \mathbb{E}[|X|^q] < \infty\}$

$$\hat{T}(a,0) = \frac{1}{N} \sum_{k=1}^{N} \psi(aX_k) \sim a^\lambda \text{ as } a \to 0$$
Numerical demonstration

Characteristic function of stable law at the origin

\[ \phi(u) = \Re \phi(u) = \exp(-|u|^\alpha) \approx 1 - |u|^\alpha + \ldots \quad (|u| \to 0) \]

\[ \alpha = 0.6 \]
Numerical demonstration

Characteristic Function

\[ \phi(t) \]

\[ \alpha = 0.6 \]

Wavelet Transform

\[ T(a, b) = \int \Re \phi(t) \cdot \frac{1}{a} \psi \left( \frac{t - b}{a} \right) dt \]

Estimation of scaling exponent

Estimated Local Holder Exponent = 0.60232

LR of local maxima line gives \( \alpha(0) = -Q_{\text{min}} = -0.60232 \)
Practical Considerations

- **Choose a wavelet**
  - With high enough regularity \((N > \lambda)\)
  - With **real positive** Fourier transform
    (ex: even derivatives of Gaussian kernel)

- **Cutoff scales** \(J_0 < j < J_1\)
  - \(j = \log(a)\): logarithmic scale
  - Shannon argument on max \(\{x_i\}\): lower bound \(J_0\)
  - Body / Tail frontier: upper bound \(J_1\)

- **Interpretation of estimator**:
  - Weight-average of samples with weight \(\Psi(aX)\)
  - Shift weights out to large samples by scaling \(a \to 0\)

\[
\frac{1}{N} \sum_{k=1}^{N} \Psi(aX_k) \approx a^\lambda \quad \text{as } a \to 0
\]
Cutoff scales

Ex: Hybrid distribution (Gamma body and stable tails)

- (for $x \geq \delta$)
  - $x \sim \alpha$-stable ($\beta=1$),
  - $\mathbb{E} |x|^r = \infty$, $r \geq \alpha$

- (for $x < \delta$)
  - $x \sim \Gamma(\gamma)$
  - $\mathbb{E} |x|^r = \infty$, $r \leq -\gamma$

$J_0$: Sampling    $J_1$: Body/Tail

Log $T(a,0)$  Log $a$
Competing for stable parameter

Alpha-stable Laws:
• compare with Koutrouvelis’80 and McCullogh’86 are parametric (stable distribution)
• non-parametric wavelet based estimator is
  • competitive
  • especially for intermediate to small $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.2</th>
<th>0.6</th>
<th>1</th>
<th>1.4</th>
<th>1.8</th>
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</thead>
<tbody>
<tr>
<td>Wavelet based</td>
<td>$0.196 \pm 0.007$</td>
<td>$0.58 \pm 0.018$</td>
<td>$1.0 \pm 0.035$</td>
<td>$1.46 \pm 0.066$</td>
<td>$1.74 \pm 0.02$</td>
</tr>
<tr>
<td>$\hat{\alpha}$ (Koutrouvelis)</td>
<td>ND</td>
<td>$0.60 \pm 0.007$</td>
<td>$1.0 \pm 0.009$</td>
<td>$1.403 \pm 0.013$</td>
<td>$1.80 \pm 0.012$</td>
</tr>
<tr>
<td>$\hat{\alpha}$ (McCullogh)</td>
<td>$0.59 \pm 0.0018$</td>
<td>$0.605 \pm 0.009$</td>
<td>$1.0 \pm 0.009$</td>
<td>$1.40 \pm 0.016$</td>
<td>$1.80 \pm 0.022$</td>
</tr>
</tbody>
</table>
Competing for Pareto parameter

1/Gamma Laws:
- Pareto
- Koutrouvelis’80 and McCulloch’86 are parametric (stable distribution)
- non-parametric wavelet based estimator is superior

<table>
<thead>
<tr>
<th>γ</th>
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<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelet based</td>
<td>0.204 ± 0.007</td>
<td>0.395 ± 0.008</td>
<td>0.589 ± 0.015</td>
<td>0.793 ± 0.03</td>
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<tr>
<td>$\hat{\alpha}$ (Koutrouvelis)</td>
<td>ND</td>
<td>0.433 ± 0.006</td>
<td>0.56 ± 0.007</td>
<td>0.67 ± 0.009</td>
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<tr>
<td>$\hat{\alpha}$ (McCulloch)</td>
<td>0.513 ± 0.000</td>
<td>0.514 ± 0.000</td>
<td>0.583 ± 0.009</td>
<td>0.72 ± 0.013</td>
</tr>
</tbody>
</table>
Model identification

...through scaling of moments
Identify the process

- One of these signals is a stable Levy flight,
- ...the other is a multiplicative cascade.
- Which is which?

Self-similar Levy flight: \[ \mathbb{E}[|L(t + \delta) - L(t)|^q] \sim \delta^{qH} \]

Multiplicative Cascade: \[ \mathbb{E}[|M(t + \delta) - M(t)|^q] \sim \delta^\tau(q) \]
Wavelet transform: $a^{-\frac{1}{2}} \int x(t) \psi((t-b)/a) \, dt$

Challenge: which wavelet to use.
Estimating $S(a, q) := \mathbb{E}|T(a, b)|^q$

Challenge: which orders to use.
Estimate of $\tau$ from

$$S(a, q) := \mathbb{E}|T(a, b)|^q \sim a^{\tau(q)}$$

Challenge: Interpretation.
The moments exist only for a few $q$. The linear $\tau(q)$ hints to a selfsimilar process (Levy flight).

The moments exist in a wide range. The non-linear $\tau(q)$ hints to a multiplicative cascade.