Capturing Market Dynamics

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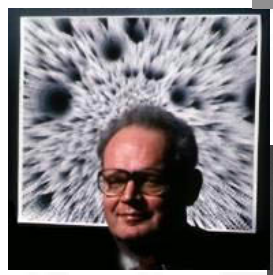
Market models

The ongoing discussion
about the role of
LRD, fBm and cascades
in Financial data

History

- Bachelier 1900
  - Computes “Barrier Option” for price modeled by Brownian motion
- Mandelbrot 1962:
  - self-similarity of Cotton prices
- Samuelson
  - *Foundations of Economic Analysis* (1947)
  - keep only minumum of simple economic relations;
  - rewrite it as a constrained optimization problem.
- Black-Scholes(’73)-Merton(’70s) (Nobel 1997)
  - Log returns are BM (returns are geometric Brownian motion)
  - Derive partial differential equation for option price

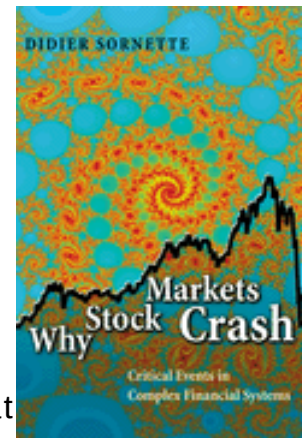


- Girsanov (’60) theorem
  - Equivalent martingale measure (pricing measure) simplifies the computation of option prices

- Engle (Nobel 2003)
  - ARCH, Cond.Duration and other time series models

Ongoing

- Estimating Fractal Dimension Of S&P 500 Using Wavelet Analysis
  - Bayraktar, Poor, Sircar 2003
- Dynamics of financial markets – Mandelbrot's cascades and beyond.
  - Muzy et al 2005
- London Stock Exchange: signs of orders obey LRD with H = 0.7.
  - This would suggest a very strong market inefficiency.
  - However, compensated for by anti-correlated fluctuations in transaction size and liquidity
    - Lillo, Farmer 2004
- Why Stock Markets Crash
  - Local logarithmic chirps
    - Sornette

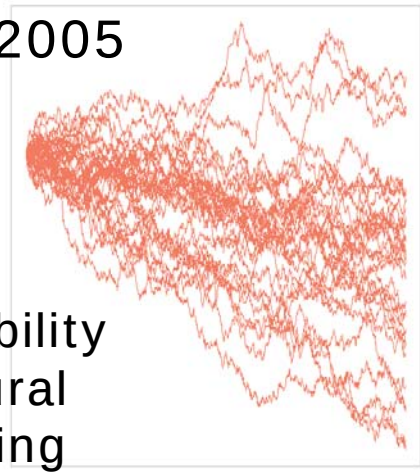


fBm and arbitrage

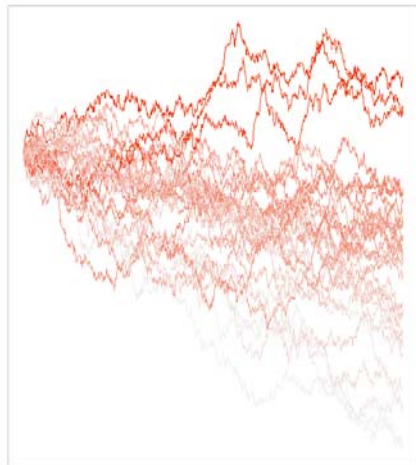
- Arbitrage strategy for Black-Scholes model driven by
  - fractional Brownian motion or
  - by a time changed fractional Brownian motion,
  - when the volatility is stochastic.
    - Bayraktar, Poor 2005
  - fBm is not semi-martingale, nor a Markov process
  - Black-Scholes differential equation not well-defined
  - No equivalent martingale measure for fBm, and therefore in a frictionless market where continuous trading is possible there exist arbitrage strategies.

fBm and Stochastic Calculus

- Girsanov formula, Ito formula for fBm
  - Conditional expectation w.r.t. fBm
    - Gripenberg and Norros.
    - Says that you need 1 minute of past to predict a minute of future
  - On Girsanov and equivalent martingale
    - Norros et al 2005
    - Decreusefond and Ustunel 2005



Paths of a Brownian motion with negative drift



The same paths reweighted according to the Girsanov formula

Illustrating Girsanov's equivalent probability measure: Left: BM with drift under natural probability. Right: paths colored according to Girsanov's equiv. probab. Measure [under which the trend is removed]

Conclusion: we need

- Estimation procedures

For LRD and multifractal scaling

- new improved models

Multifractal estimation

Large Deviations

Multifractal formalism

Legendre transform

Counting via Large Deviations

Recall

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$E_a := \{t \,:\, \alpha(t) = a\}$$

- Notation:
  - Number of dyadic intervals with exponent ~ a:

$$N_{n,\delta}(a) := \#\{(\epsilon_1 \dots \epsilon_n) \,:\, a-\delta \le \alpha_n(\epsilon_1 \dots \epsilon_n) < a+\delta\}.$$

  - Partition sum: a microscope inspired by LDP

$$S_n(q) := \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q = \sum_{\epsilon_1 \dots \epsilon_n} |2^n|^{q\alpha_n(\epsilon_1 \dots \epsilon_n)}.$$

  - Assume powerlaws:

$$N_{n,\delta}(a) \sim 2^{nf(a)} \qquad S_n(q) \sim 2^{-n\tau(q)}$$

  - Typically (LDP)

$$\boxed{f(a) = \inf_q (qa - \tau(q))}$$

Multifractal estimation via wavelets

- Set indicator of oscillation to be wavelet coefficient:

$$I(\epsilon_1 \ldots \epsilon_n) := [t_n, t_n + 1/2^n)) \quad t_n := \sum_{k=1}^{n} \epsilon_k / 2^k$$

$$\Delta I_n(\epsilon_1 \ldots \epsilon_n) = d_{2^n t_n, -n}$$

- Log-log regression of S yields estimate

$$S_n(q) := \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q = \sum_j |d_{j,-n}|^q \sim 2^{-n\tau(q)}$$

- From before: $\tau(2)$ estimates 2H-1 for fBm
  - ...as it should since $\tau(q) = qH-1$

fBm simulation

Fractional ARIMA

- Recall the definition which allows for sequential simulation

$$X_i = \Delta^{-d} \epsilon_i, \quad i \geq 1,$$

- ...and has simpler spectral density than fGn

$$f(\lambda) = \frac{1}{2\pi}\left(2\sin\frac{\lambda}{2}\right)^{-2d} \sim \frac{1}{2\pi}|\lambda|^{-2d} \text{ as } \lambda \to 0.$$

Fast exact Synthesis of fBm

- To produce exact correlation structure need to decompose auto-correlation matrix

- “Naïve” Decomposition requires O(N3)
  - Allows only sample size of a few hundred
- Cholesky: O(N2)
  - Restrict to synthesis length of ~1000
- Circulant method:
  - Embed correlation matrix in circulant matrix
  - Decomposition is then exactly the FFT
  - Allows to synthesize fGn of ~1’000’000
- Large simulated traces needed for “trading time models” (see next)

Multifractal Subordination

Processes with
multifractal oscillations

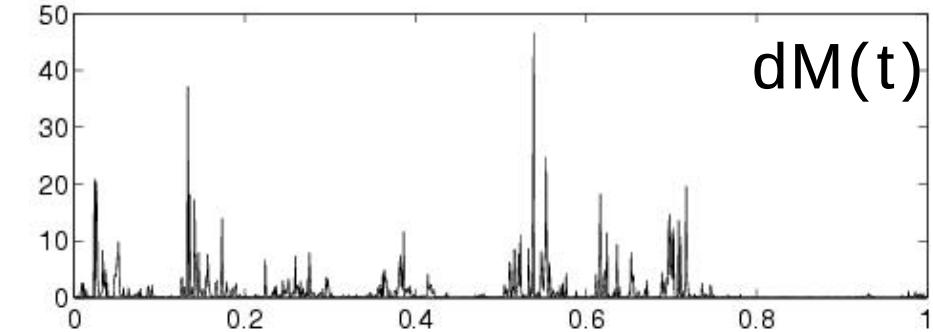
Multifractal time warp

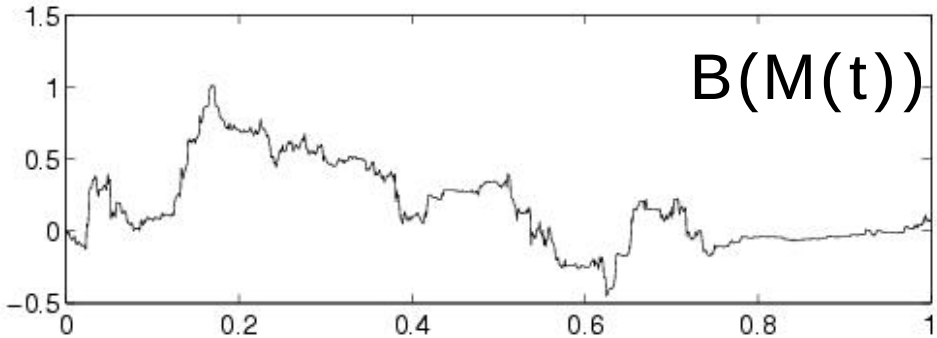
$B_H(M(t))$: B_H fBm, dM independent measure

A versatile model

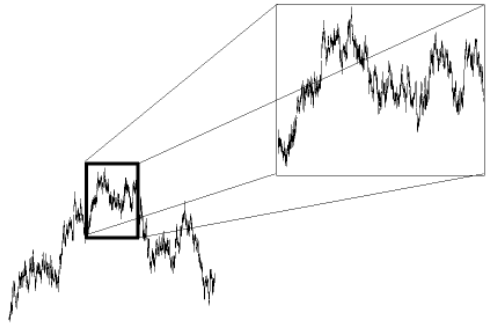
- M(t): Multifractal
  Time change
  Trading time

- B: Brownian motion
  Gaussian fluctuations





Hölder regularity



- Levy modulus of continuity:
  - With probability one for all t

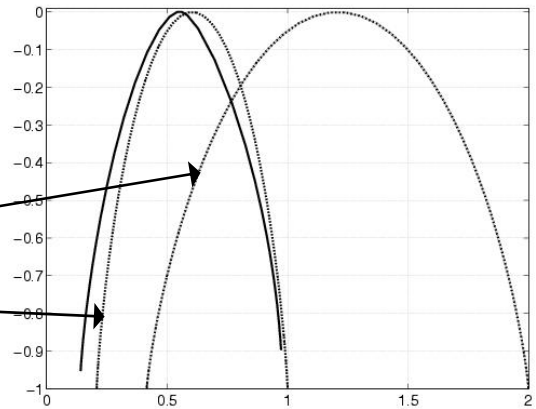
$$|B_H(t+\delta) - B_H(t)| \simeq |\delta|^H$$

  - Thus, exponent gets stretched:

$$|B_H(M(t+\delta)) - B_H(M(t))| \simeq |M(t+\delta) - M(t)|^H \simeq |\delta|^{H\alpha(t)}$$

  - and spectrum gets squeezed:

$$\dim E_a[B_H(M)] = \dim E_{a/H}[M]$$



Auto-Correlation of B(M(t))

- Conditional on M: from auto-correlation of B

$$\mathbb{E}[B(M(t))B(M(s))|M] = (\sigma^2/2)[M^{2H}(t) + M^{2H}(s) - |M(t) - M(s)|^{2H}]$$

- For the increments $X(t)+B(M(t+1))-B(M(t))$

$$\begin{aligned}\mathbb{E}[X(t)X(s)|M] \;=\; & (\sigma^2/2)\Big([M(t+1) - M(s)]^{2H} - [M(t) - M(s)]^{2H} \\ & -[M(t+1) - M(s+1)]^{2H} + [M(t) - M(s+1)]^{2H}\Big)\end{aligned}$$

- Assume $H = 1/2$ and $\mathbb{E}[(M(t)] = t$. Then

$$\mathbb{E}[B(M(t))B(M(s))] = \sigma^2 \min(\mathbb{E}[(M(s)], \mathbb{E}[(M(t)]) = \sigma^2 \min(s,t)$$

$$\mathbb{E}[X(t+k)X(t)] = \mathbb{E}[M(k+1)-2M(k)+M(k-1)] = 0$$

- Uncorrelated, 2nd order stationary, but not Gaussian
- Dependence of higher order through M(t)
- Blind spot of spectral analysis

Estimation: Wavelets decorrelate

Decorrelation: (with P. Goncalves)

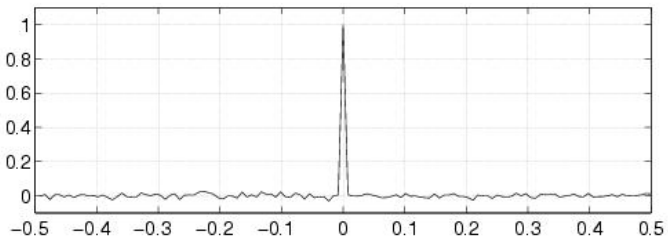
$$
\begin{aligned}
&\mathbb{E}[W_{jk}W_{jm}] \\
&\overset{1}{=} \mathbb{E}\int\int \psi_{jk}(t)\psi_{jm}(s)B(M(t))B(M(s))dtds \\
&\overset{2}{=} (\sigma^2/2)\int\int \psi_{jk}(t)\psi_{jm}(s)\mathbb{E}\left[M^{2H}(t)+M^{2H}(s)-|M(t)-M(s)|^{2H}\right]dtds \\
&\overset{3}{=} -(\sigma^2/2)\int\int \psi_{jk}(t)\psi_{jm}(s)\mathbb{E}\left[|M(t-s)|^{2H}\right]dtds \\
&\overset{4}{=} -(\sigma^2/2)\int |2^j\tau|^{T(2H)}\gamma_\psi(\tau-(k-m))\,d\tau \\
&\overset{5}{=} -(\sigma^2/2)|2^j|^{T(2H)}\int \frac{|\Psi(\nu)|^2}{|\nu|^{T(2H)+1}}\,e^{-i2\pi\nu(k-m)}d\nu, \\
&\overset{6}{\sim} O\left(|k-m|^{T(2H)+1-2N}\right) \qquad (|k-m|\to\infty)
\end{aligned}
$$

(1) $W_{jk}=\int\psi_{jk}(t)B(M(t))dt$
(2) $\mathbb{E}[B(u)B(v)]=(\sigma^2/2)[u^{2H}+v^{2H}-|u-v|^{2H}]$
(3) $\int\psi(s)ds=0=\int\psi(t)dt$,
(4) $\psi_{jk}(t)=2^j\int\psi(t/2^j-k)$, $\tau=2^{-j}(t-s)$, $\gamma_\psi(\tau)=\int\psi(t)\psi(t+\tau)dt$, $\mathbb{E}[M(h)]=h^{T(q)}$
(5) Parceval
(6) $\Psi(\nu)\sim|\nu|^N$

Multifractal Estimation for B(M(t))

- Weak Correlations of Wavelet-Coefficients:
  (with P. Goncalves)

Haar



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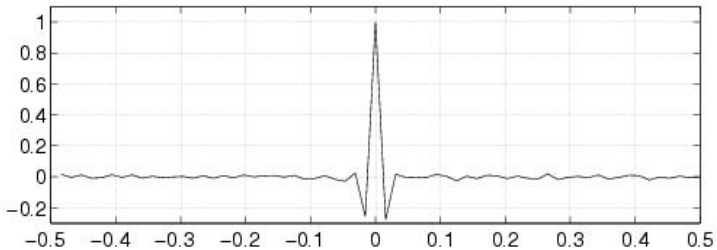


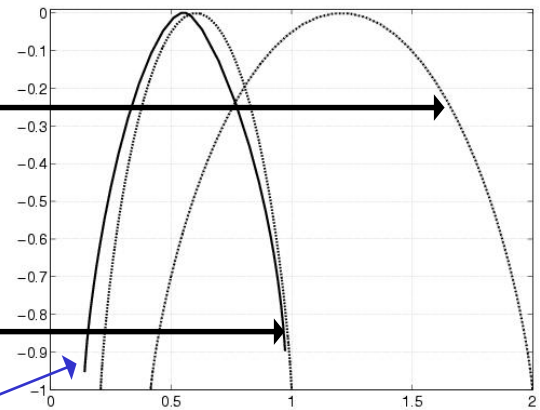
- Improved estimator due to weak correlations
- Multifractal Spectrum

$M(t+s) - M(t) \sim s^{a(t)}$

$B(t+u) - B(t) \sim u^{H} \quad (\forall\ t)$

$\rightarrow$

$B(M(t+s)) - B(M(t)) \sim s^{H*a(t)}$



Estimation

Multifractal time in Zoology

- From Random Walk to Multifractal Random Walk in Zooplankton Swimming Behavior
  - Seuront, Schmitt, Brewer, Strickler and Sami 2004

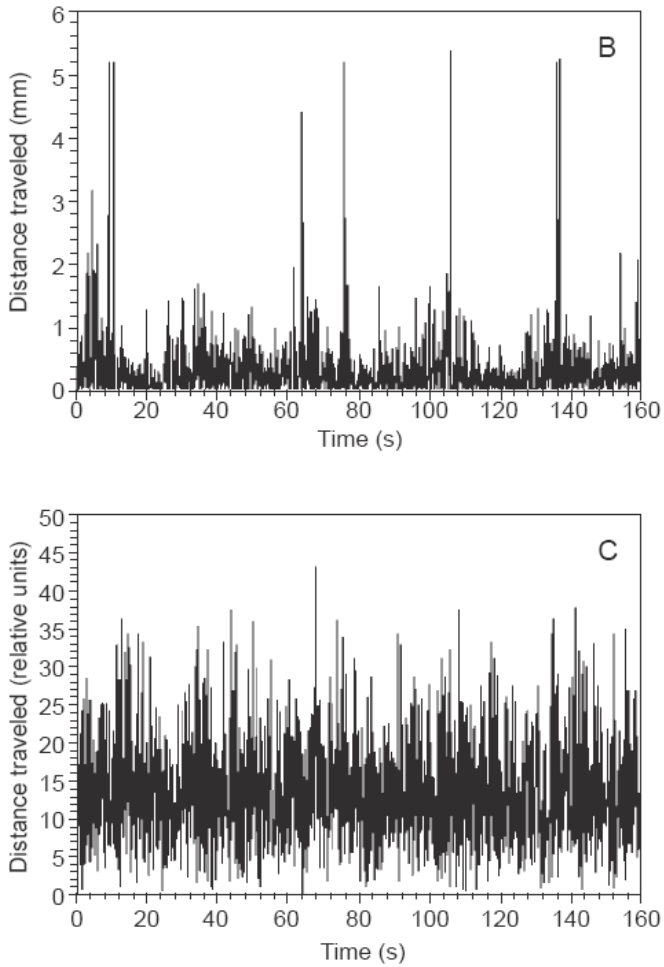


Fig. 1. Examples of distance traveled by *Temora longicornis* (12.5-Hz resolution; A) and *Daphnia pulex* (10-Hz resolution; B), compared to the distance traveled in the case of pure random motion (C).

Tail estimation

And diverging moments

Diverging Moments

Diverging moments: $\mathbb{E}|X|^q = \infty$

They bear on...

- Estimation of tails: $P[|X| > x] \sim x^{-\alpha}$
- Estimators per se:
  - Bias $(X_1^2 + ...X_n^2)/n$
  - Asymptotic normality

Theory

Tails

- Let $\lambda > 0$.

$$\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda}$$

$$\Longleftrightarrow$$

$$\boxed{P[|X| > 1/u] \overset{u \to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda}$$

- "Finiteness of Moments and Tails go together"

Characteristic function 101

- Characteristic function:

$$\phi(u) = \mathbb{E}[\exp(iuX)]$$

- Char Fct and Moments 101: If $E|X|^n$ is exists then

$$\phi^{(n)}(0) = i^n \mathbb{E}[X^n]$$

- Vice versa: If ϕ has 2p derivates then $E|X|^{2p}$ exists

Characteristic function 102

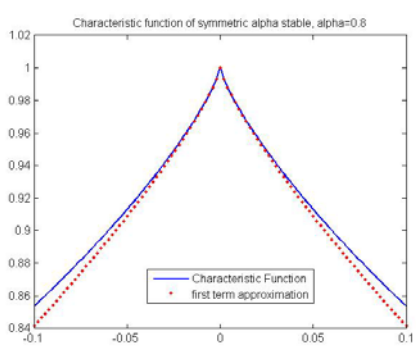
- Char Fct and Moments 102: (Tauberian Theorem)
  For $0<\lambda<2$

$$\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda} \quad \Longleftrightarrow$$

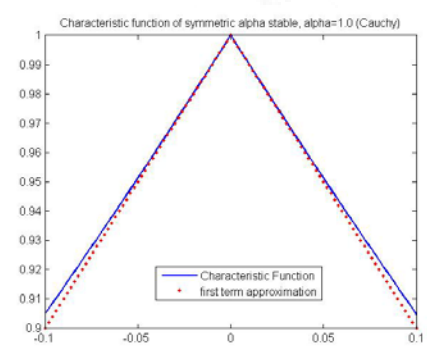
$$\boxed{\mathcal{R}e\ \phi(u) - 1 \overset{u\to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda}$$

- Example: symmetric stable laws (moments up to α)

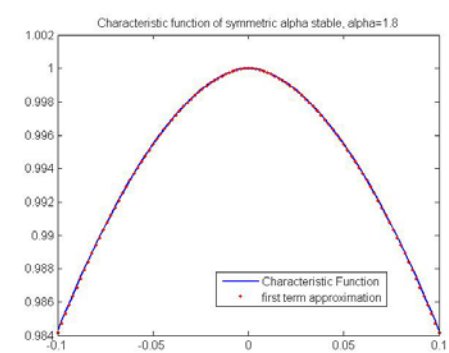
$$\phi(u) = \mathcal{R}e\ \phi(u) \ = \ \underline{\exp(-|u|^\alpha)} \simeq \underline{1 - |u|^\alpha} + \ldots \qquad (|u| \to 0)$$



$\alpha = 0.8$



$\alpha = 1$: Cauchy



$\alpha = 1.8$

Extension to orders > 2

- Kawata ('72) / Lukacs ('83) / Ramachandran ('69):
  - Let $2p<\lambda\leq 2p+2$ with integer p.

$$\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda} \quad \Longleftrightarrow$$

$$\boxed{\begin{array}{l} \mathbb{E}[|X|^{2p}] < \infty, \text{ i.e. } \phi^{2p}(0) \text{ exists, and} \\ \mathcal{R}e\ \phi(u) - \sum_{k=1}^{p} \frac{(-1)^k}{(2k)!}\mathbb{E}[X^{2k}]u^{2k} \overset{u\to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda \end{array}}$$

Estimating the Regularity of ϕ

- Motivation:
  - exact regularity of ϕ at zero provides the cutoff value for finite moments
- Microscope for regularity: Wavelet transform T

$$T(a,b) = < \mathcal{R}e\ \phi, \underline{\psi_{a,b}} > = \int \mathcal{R}e\ \phi(s) \cdot \underline{\frac{1}{a}\psi\left(\frac{s-b}{a}\right)} ds$$

- Simplified regularity theorem: Assume
  - Wavelet regularity N>λ:
  - Hölder polynomial P_ϕ of c $\int t^k \psi(t) dt = 0$ for $0 \le k < N$
  - *Transform T(a,b) is maximal at b=0*
  - Then

$$\boxed{\begin{aligned} & \mathcal{R}e\ \phi(u) - P_\phi(u) \overset{u \to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda \\ \Leftrightarrow \quad & T(a,0) \overset{a \to 0}{=} O(|a|^r) \quad \text{for all } r < \lambda \end{aligned}}$$

Simplified Proof

- If (1) $\phi(u) - P_\phi(u) \overset{u \to 0}{=} O(|u|^r)$

  and if (2) the wavelet ψ is supported on [0,1]
- then $T(a,0) \overset{a \to 0}{=} O(|a|^r)$

$$
\begin{aligned}
|T(a,0)| = \left| < \phi, \psi_{a,0} > \right| &= \left| \frac{1}{a} \int_0^a \phi(s)\, \psi(s/a)\, ds \right| \\
&\overset{(2)}{=} \left| \frac{1}{a} \int_0^a (\phi(s) - P_\phi(s))\, \psi(s/a)\, ds \right| \\
&\leq C \cdot \frac{1}{a} \int_0^a |s|^r\, |\psi(s/a)|\, ds \\
&\overset{(1)}{\leq} C \cdot a^r \frac{1}{a} \int_0^a |\psi(s/a)|\, ds \\
&\leq C \cdot a^r \cdot \int_{\mathbb{R}} |\psi(s)|\, ds
\end{aligned}
$$

$\int t^k \psi(t) dt = 0$ for $0 \leq k < N$

Wavelet Transform of ϕ

- Fourier transform:

$$(i) \quad \Psi_{a,b}(x) = \frac{1}{a}\int \psi(\frac{s-b}{a})e^{isx}ds = \int \psi(u)e^{i(au+b)x}du = e^{ixb}\Psi(ax)$$

- Parseval:

$$(ii) \quad T(a,b) =< \mathcal{R}e\ \phi, \psi_{a,b} >= \mathcal{R}e < F, \Psi_{a,b} >= \mathcal{R}e\,\mathbb{E}[\Psi_{a,b}(X)]$$

- Assume: Fourier Transform Ψ of ψ is real positive.
  - then:

$$|T(a,b)| \overset{(ii)}{\leq} \mathbb{E}[|\Psi_{a,b}(X)|] \overset{(i)}{=} \mathbb{E}[|\Psi(aX)|] = |T(a,0)|$$

  - in other words: $|T(a,b)|$ *maximal at b=0*

- Ex:

$$\Psi(u) = u^{2n}\exp(-u^2) \geq 0$$
$$\psi(x) = (-1)^n(\tfrac{d}{dx})^{2n}\exp(-x^2)$$

Wavelet Transform of ϕ

- Summary: Assume: Ψ is real positive
  - Then |T(a,b)| maximal at b=0
  - ... and

$$(iii) \quad T(a,0) = \mathcal{R}e\,\mathbb{E}[\Psi_{a,0}(X)] = \mathcal{R}e\,\mathbb{E}[\Psi(ax)] = \mathbb{E}[\Psi(ax)]$$

- Recall equivalent conditions for $0<\lambda<2$:

$$(1) \quad \mathcal{R}e\ \phi(u) - 1 \overset{u\to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda$$

$$(2) \quad T(a,0) \overset{a\to 0}{=} O(|a|^r) \quad \text{for all } r < \lambda$$

- → estimate regularity of Re(ϕ) by the powerlaw

$$\boxed{|T(a,0)| = \mathbb{E}[\Psi(aX)] \sim a^{\lambda}}$$

Implementation
and
Performance

Numerical Implementation

$$|T(a,0)| = \mathbb{E}[\Psi(aX)] \sim a^{\lambda}$$

The estimator of T(a,0) of ϕ is

- ...simple:

$$\hat{T}(a,0) = \hat{\mathbb{E}}[\Psi(aX)] = 1/N \sum_{k=1}^{N} \Psi(aX_k)$$

- ...unbiased
- ...non-parametric!

- Estimation of critical order $\lambda = \sup\{q: E[|X|^q]<\infty\}$

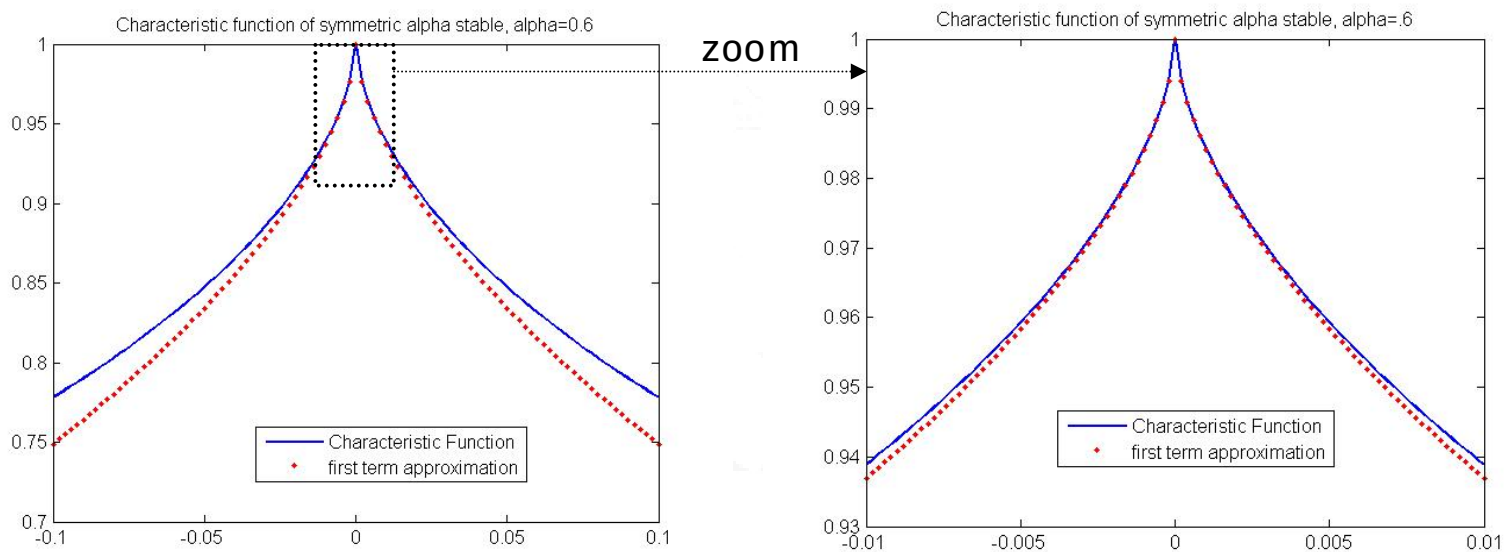
$$1/N \sum_{k=1}^{N} \Psi(aX_k) \simeq a^{\lambda} \quad \text{as } a \to 0$$

Numerical demonstration

Characteristic function of stable law at the origin

$$\phi(u) = \mathcal{R}e\ \phi(u) \ = \ \underline{\exp(-|u|^\alpha)} \simeq \underline{1 - |u|^\alpha} + \ldots \qquad (|u| \to 0)$$

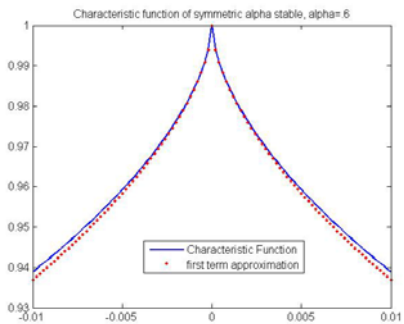
$$\alpha = 0.6$$



Numerical demonstration

Characteristic Function

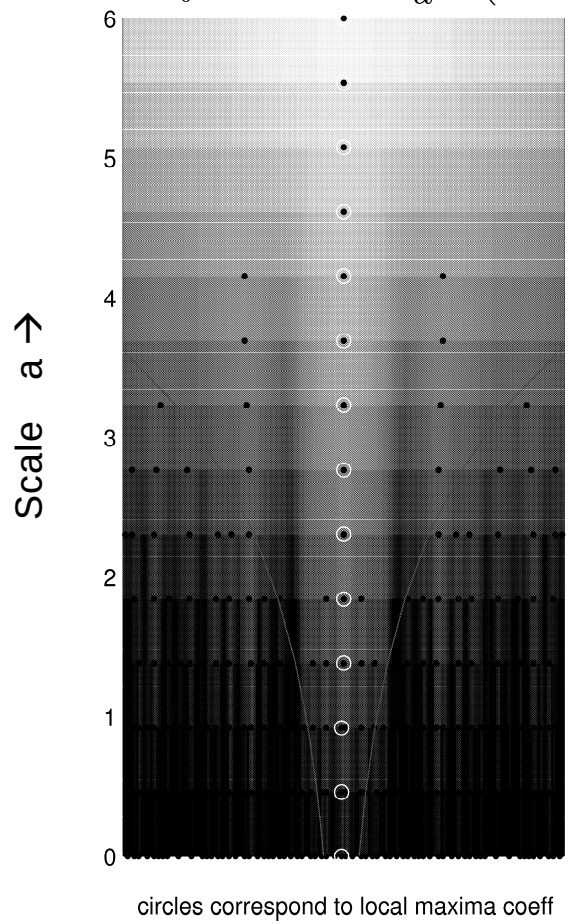
$$\phi(t)$$



$$\alpha = 0.6$$

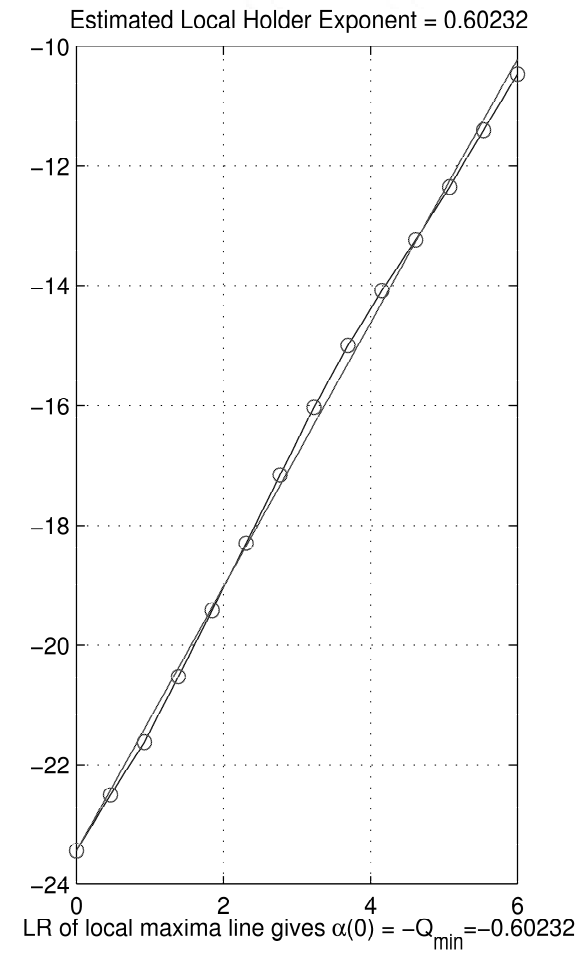
Wavelet Transform

$$T(a,b) = \int \mathcal{R}e\ \phi(t) \cdot \frac{1}{a}\psi\left(\frac{t-b}{a}\right) dt$$



circles correspond to local maxima coeff

Estimation of scaling exponent



Practical Considerations

$$1/N \sum_{k=1}^{N} \Psi(aX_k) \simeq a^{\lambda} \quad \text{as } a \to 0$$

- Choose a wavelet
  - With high enough regularity (N>λ)
  - With real positive Fourier transform
    (ex: even derivatives of Gaussian kernel)

- Cutoff scales $J_0 < j < J_1$
  - j=log(a): logarithmic scale
  - Shannon argument on max $\{x_i\}$: lower bound J_0
  - Body / Tail frontier : upper bound J_1

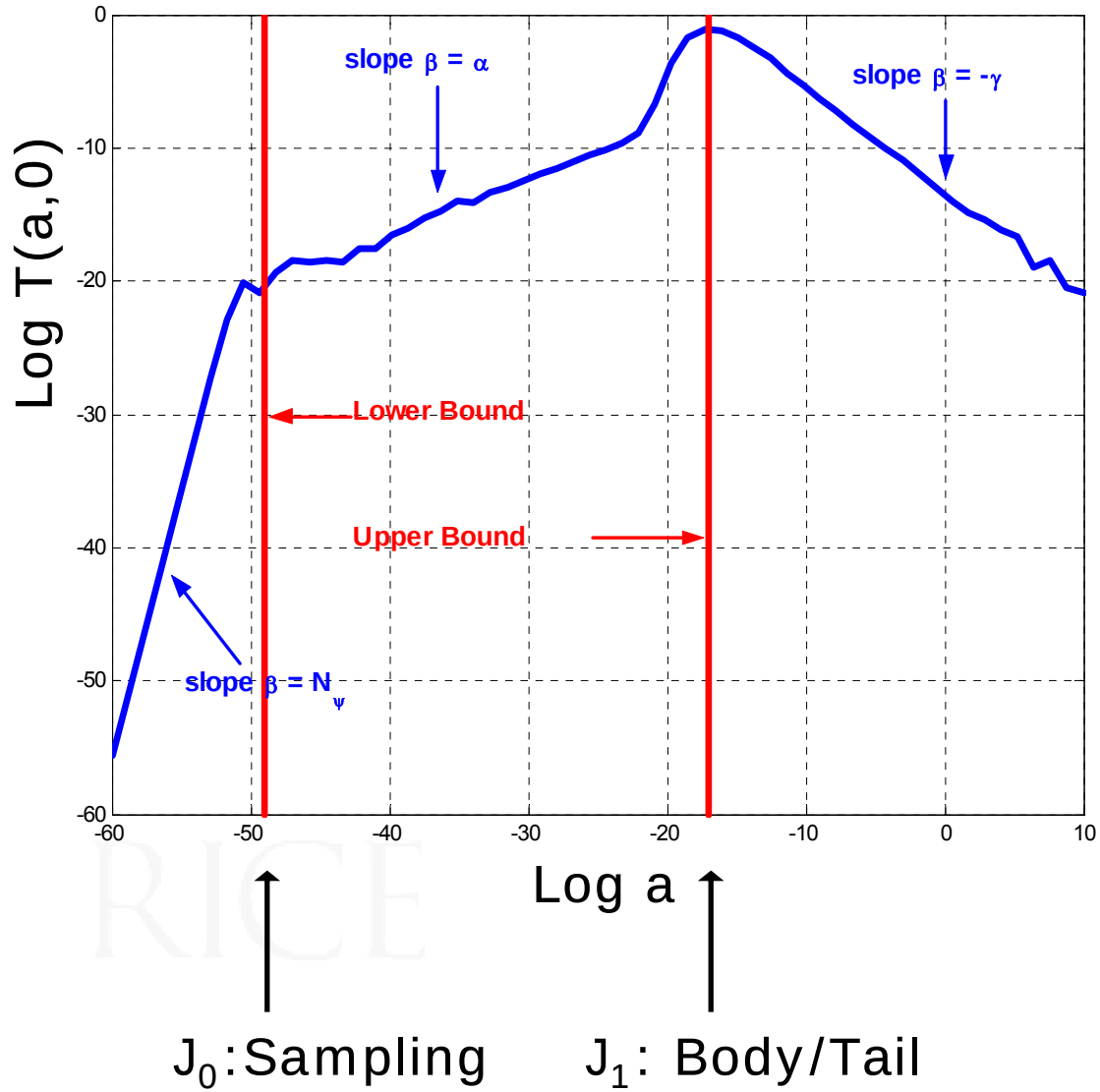
- Interpretation of estimator:
  - Weight-average of samples with weight Ψ(aX)
  - Shift weights out to large samples by scaling a→0

Cutoff scales

Ex: Hybrid distribution
(Gamma body and stable tails)

- (for $x >= \delta$)
  - $x \sim \alpha$-stable (β=1),
  - $E\,|x|^r = \infty$, $r >= \alpha$

- (for $x < \delta$)
  - $x \sim \Gamma(\gamma)$
  - $E\,|x|^r = \infty$, $r <= -\gamma$

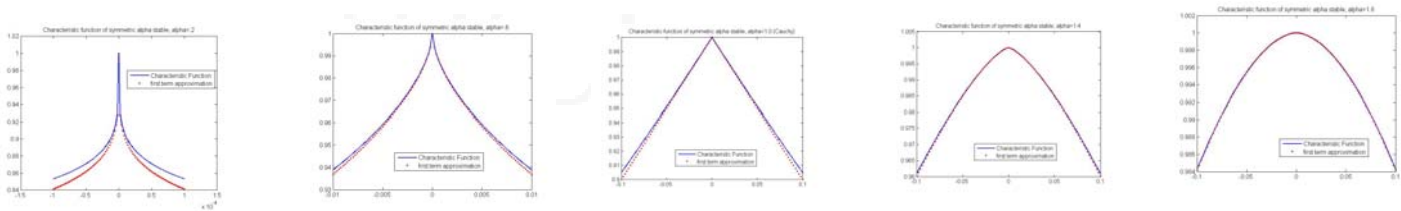


Competing for stable parameter

Alpha-stable Laws:

- compare with Koutrouvelis'80 and McCullogh'86 are parametric (stable distribution)
- non-parametric wavelet based estimator is
  - competitive
  - especially for intermediate to small a

α	0.2	0.6	1	1.4	1.8
Wavelet based	0.196 ± 0.007	0.58 ± 0.018	1.0 ± 0.035	1.46 ± 0.066	1.74 ± 0.02
$\widehat{\alpha}$ (Koutrouvelis)	ND	0.60 ± 0.007	1.0 ± 0.009	1.403 ± 0.013	1.80 ± 0.012
$\widehat{\alpha}$ (McCullogh)	0.59 ± 0.0018	0.605 ± 0.009	1.0 ± 0.009	1.40 ± 0.016	1.80 ± 0.022



Competing for Pareto parameter

1/Gamma Laws:

- Pareto
- Koutrouvelis'80 and McCullogh'86 are parametric (stable distribution)
- non-parametric wavelet based estimator is
    - superior

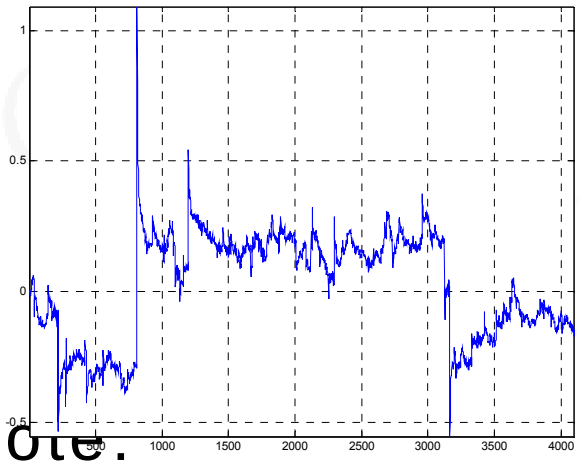
γ	0.2	0.4	0.6	0.8
Wavelet based	0.204 ± 0.007	0.395 ± 0.008	0.589 ± 0.015	0.793 ± 0.03
$\widehat{\alpha}$ (Koutrouvelis)	ND	0.433 ± 0.006	0.56 ± 0.007	0.67 ± 0.009
$\widehat{\alpha}$ (McCullogh)	0.513 ± 0.000	0.514 ± 0.000	0.583 ± 0.009	0.72 ± 0.013

Model identification

...through scaling of moments

Identify the process

- One of these signals is a stable Levy flight,
- ...the other is a multiplicative cascade.
- Which is which?

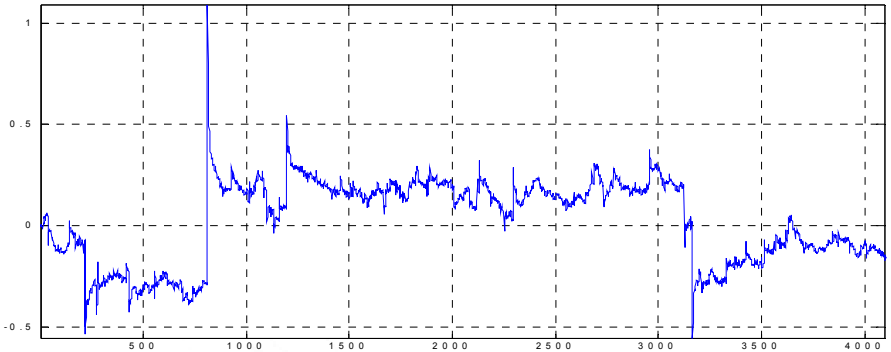


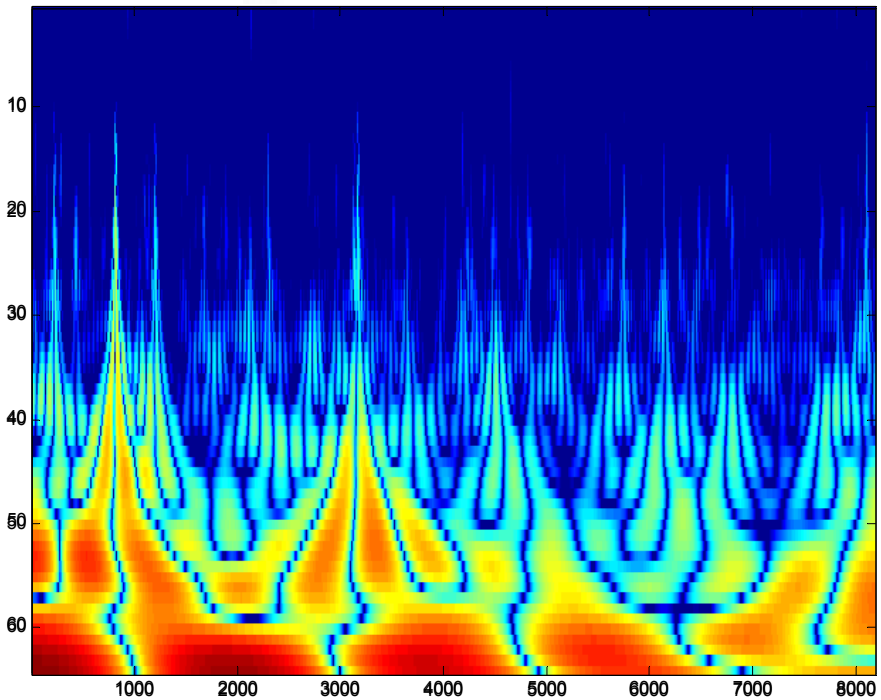
- Note:

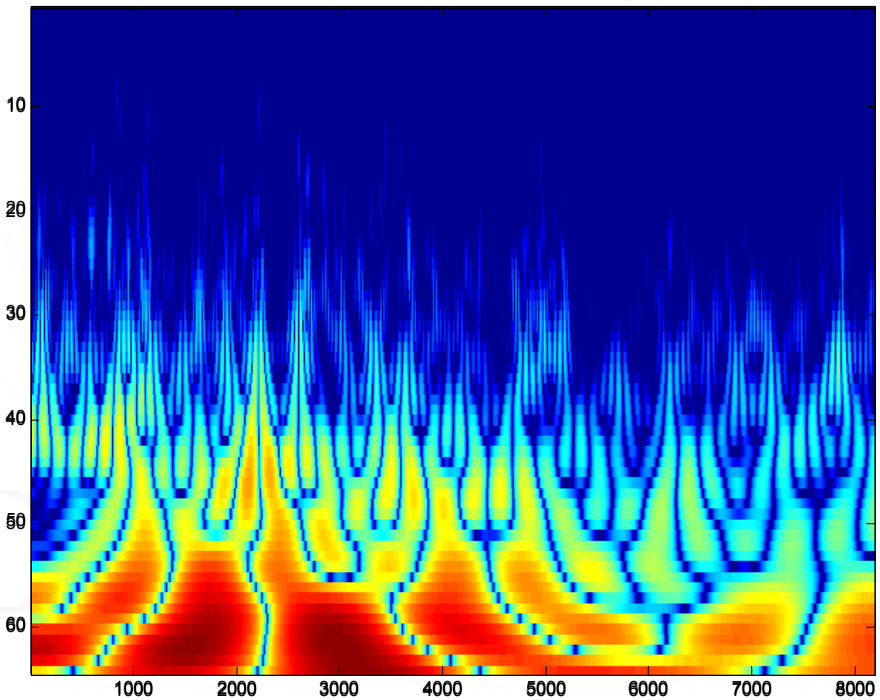
Self-similar Levy flight: $\mathbb{E}[|L(t+\delta) - L(t)|^q] \simeq \delta^{qH}$

Multiplicative Cascade: $\mathbb{E}[|M(t+\delta) - M(t)|^q] \simeq \delta^{\tau(q)}$

Wavelet transform: $a^{-1/2} \int x(t)\, \psi((t-b)/a)\, dt$

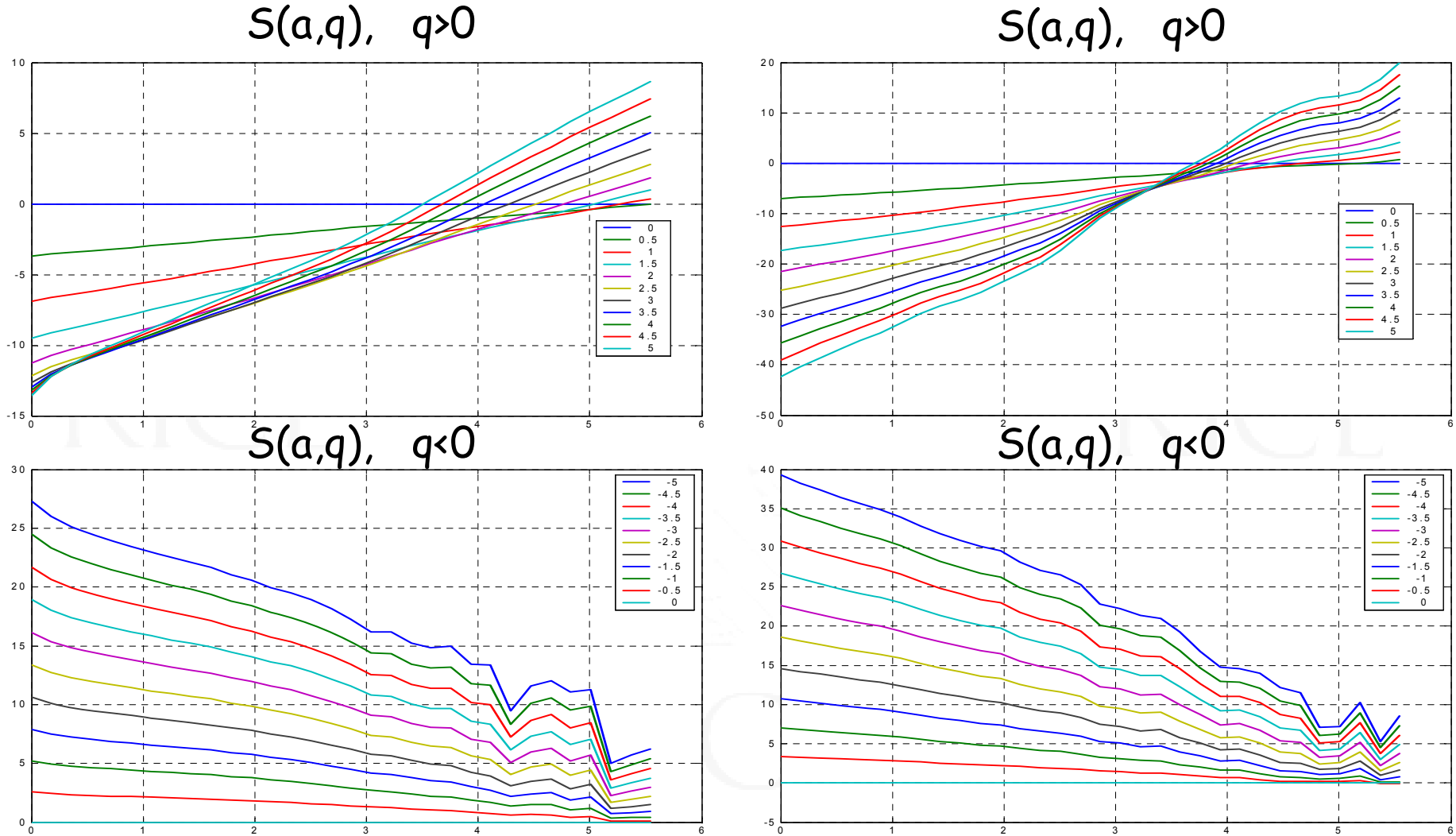






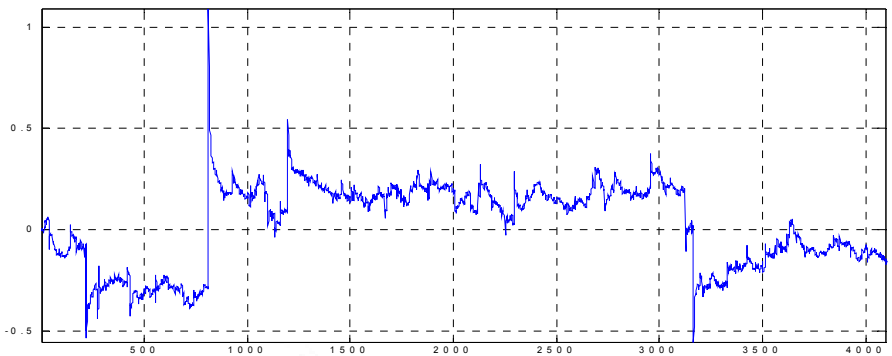
Challenge: which wavelet to use.

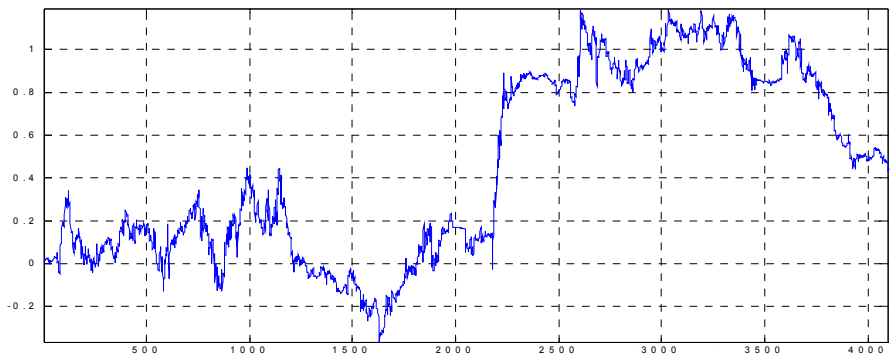
Estimating $S(a,q) := \mathbb{E}|T(a,b)|^q$

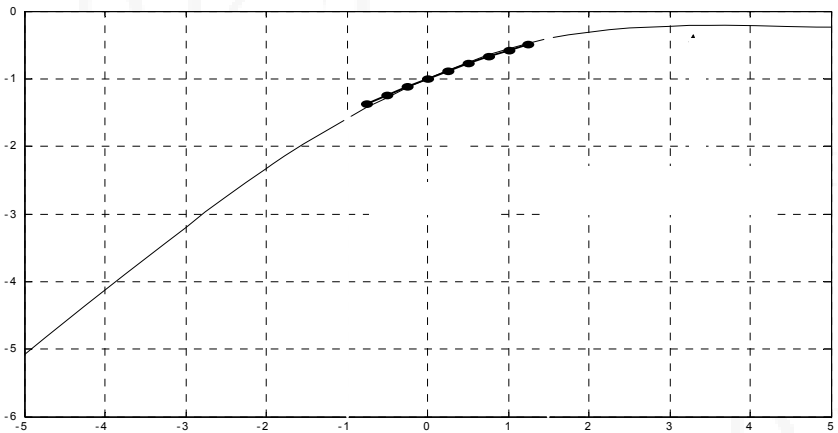


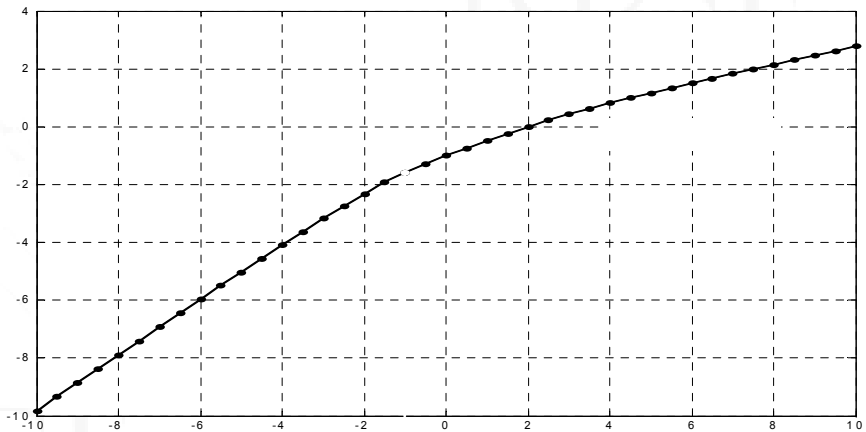
Challenge: which orders to use.

Estimate of τ from $S(a,q) := \mathbb{E}|T(a,b)|^q \simeq a^{\tau(q)}$



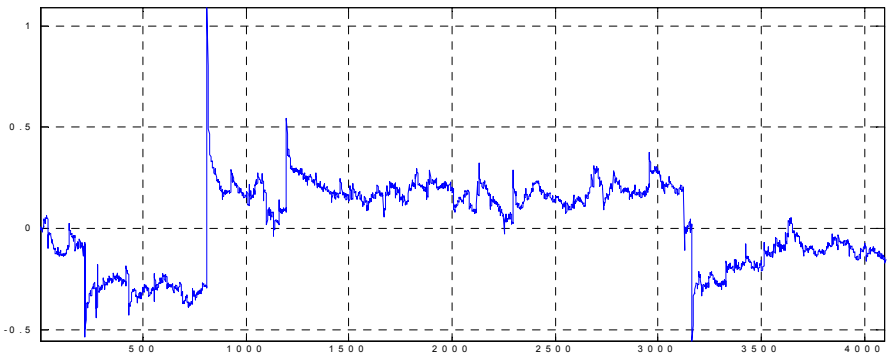


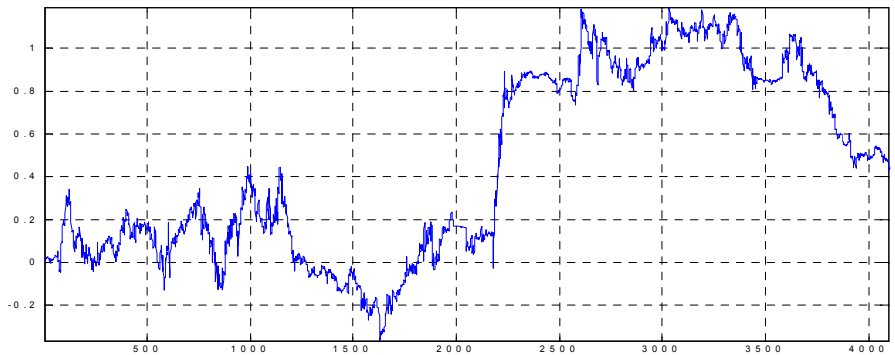


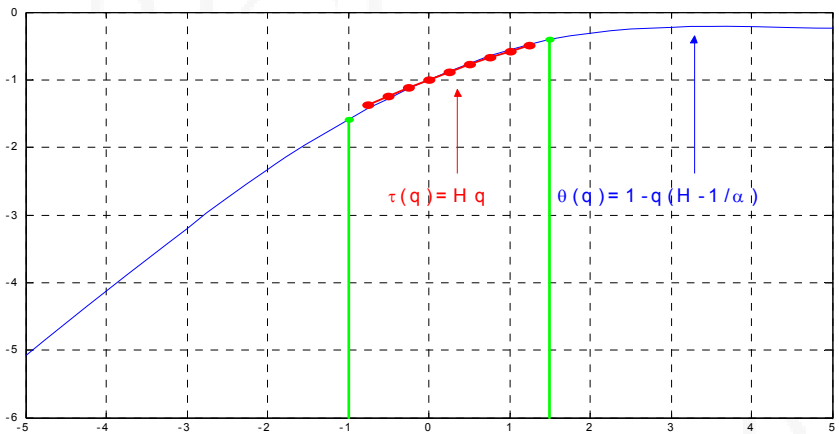


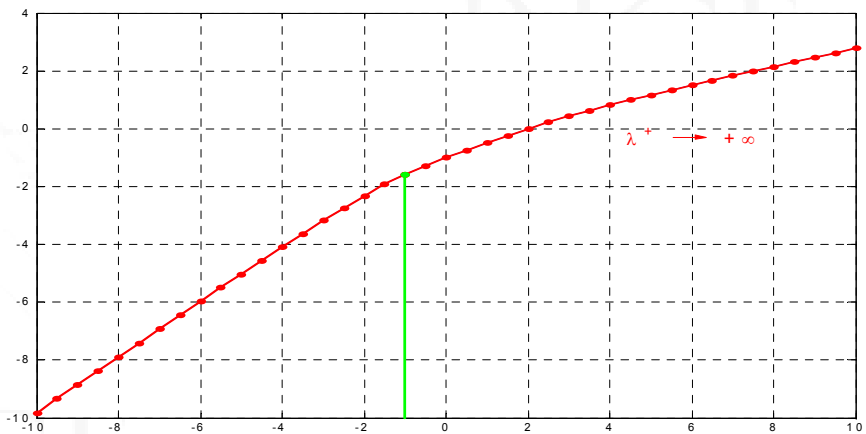
Challenge: Interpretation.

Supervised moment estimation









The moments exist only for a few q. The linear $\tau(q)$ hints to a selfsimilar process (Levy flight)

The moments exist in a wide range. The non-linear $\tau(q)$ hints to a multiplicative cascade.