Trees, Wavelets and Large Deviations

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ON-OFF limits & the small scales

- ON-OFF explains two asymptotic regimes with self-similar limits
  - Beta regime:
  - highly multiplexed slow connections → fBm
  - Alpha regime:
  - Few fast large connections → Levy stable
- However, limits are at large scales, not small.

Highly multiplexed limit is Gaussian

$$\frac{1}{m^{1/2}}\sum_{i=1}^{m}(X_i(t)-\mathsf{E}X_i(t)) \stackrel{m\to\infty}{\longrightarrow} G(t)$$

At large scales self-similar, ie: fBm

$$\frac{1}{T^H}\int_0^{Tt} G(u)du \stackrel{fdd}{\to} \sigma B_H(t)$$

$$H=\frac{3-\min(\alpha_{\text{on}},\alpha_{\text{off}})}{2}$$

Large *scale* limit K is self-similar has indep. increments and heavy tails

$$\frac{1}{T^H}\int_0^{Tt}(X_i(t)-\mathsf{E}X_i(t)) \stackrel{T\to\infty}{\longrightarrow} K_i(t)$$

Highly multiplexed becomes Levy stable motion

$$\frac{1}{m^H}\sum_i K_i(t) \stackrel{m\to\infty}{\sim} L_H(t)$$

$$H=\frac{1}{\min(\alpha_{\text{on}},\alpha_{\text{off}})}$$

Tree based models

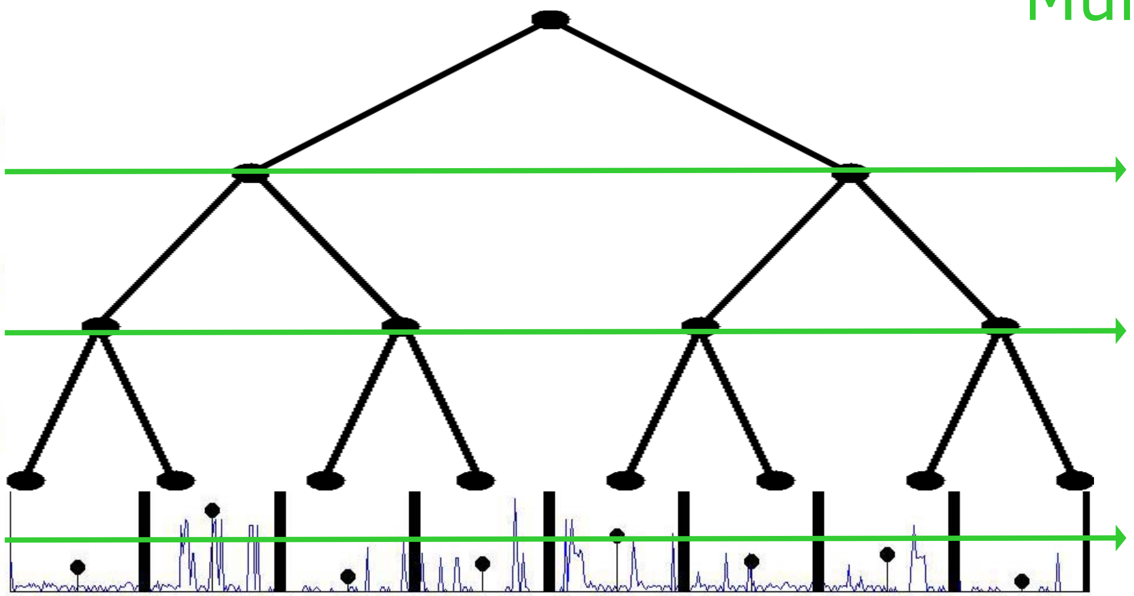
A hierarchical approach

Dyadic Multiscale Analysis

Time $\longrightarrow$

Multiscale statistics

Number of scales $j=\log_2(m)$

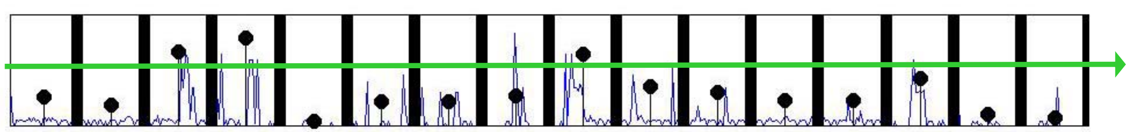


$V_1 = \mathrm{Var}\, X^{(m)}$

$V_2 = \mathrm{Var}\, X^{(m/2)}$

$V_3 = \mathrm{Var}\, X^{(m/4)}$

Flow up : $X_n^{(2m)} = X_{2n}^{(m)} + X_{2n+1}^{(m)}$



$V_j = \mathrm{Var}\, X^{(1)}$

Start at bottom with trace $X_n^{(1)}$

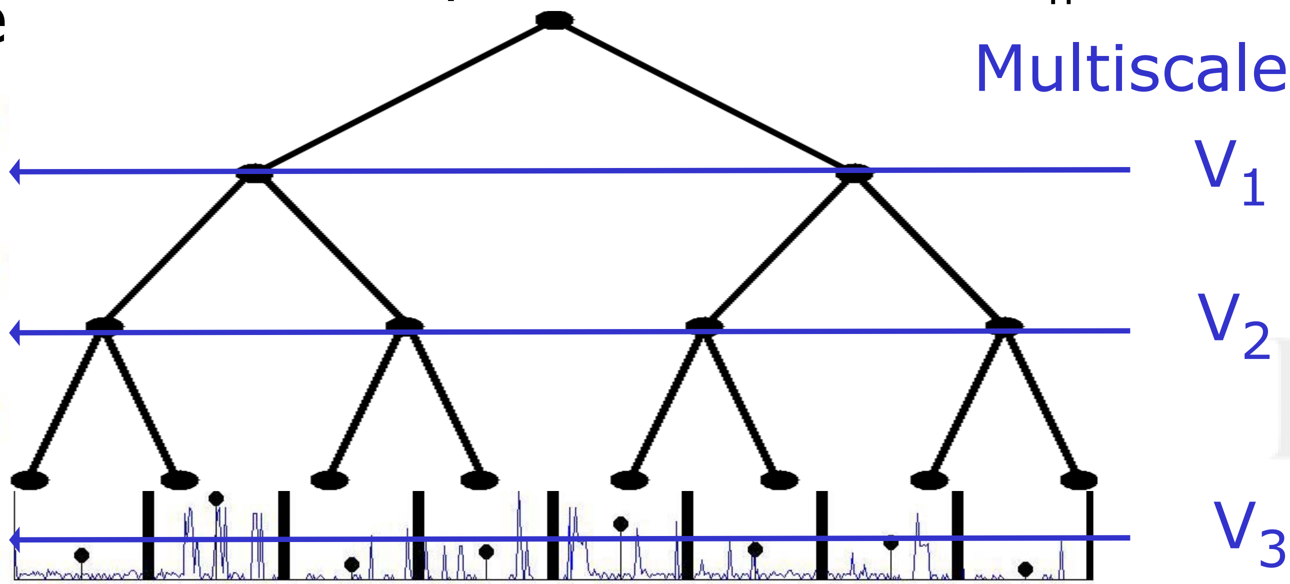
Dyadic Multiscale Synthesis

Time →

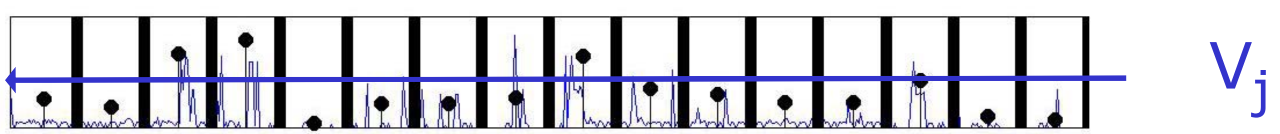
Scale ↕

Start at top with sum of all $X_n^{(1)}$

Multiscale parameters



Flow down : ¿ $X_n^{(2m)} \rightarrow X_n^{(m)}$?



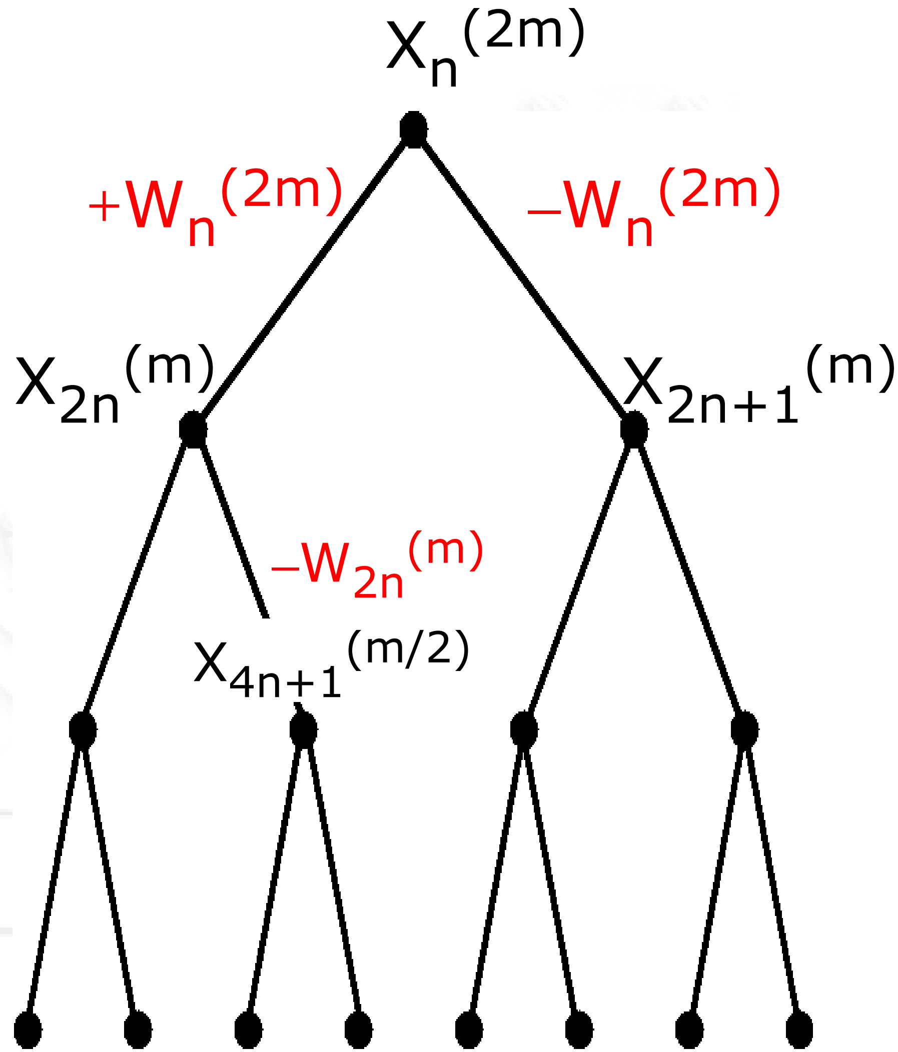
Signal: bottom nodes

Additive Innovations

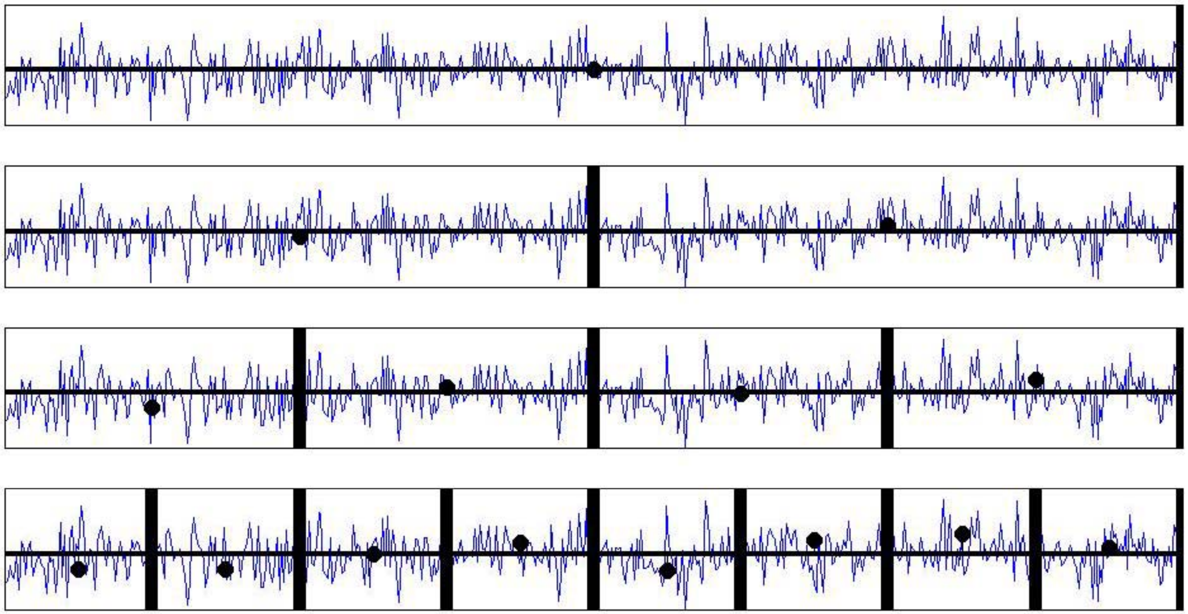
Synthesis:

- Start at root
- Flow down the tree
- Additive, independent innovations $W_n^{(m)}$
- Conservation:

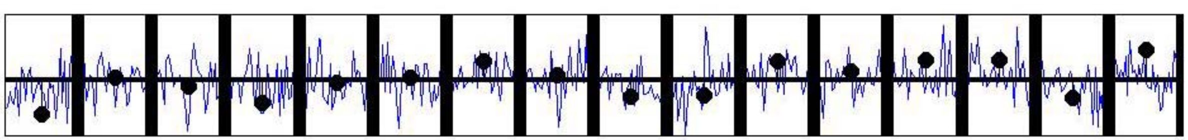
$$X_{2n}^{(m)} = (X_n^{(2m)} + W_n^{(2m)})/2$$
$$X_{2n+1}^{(m)} = (X_n^{(2m)} - W_n^{(2m)})/2$$



Additive Tree: Linear Processes



CLT: asymptotically Gaussian



Additive Innovations $W_n^{(m)} \sim \mathcal{N}(0, \sigma^2 m^{-(2H+1)})$: Model for $B_H(t)$

Multiplicative Innovations

Positive process:

- Add `small' innovation:

  $| W_n^{(m)} | < X_n^{(m)}$

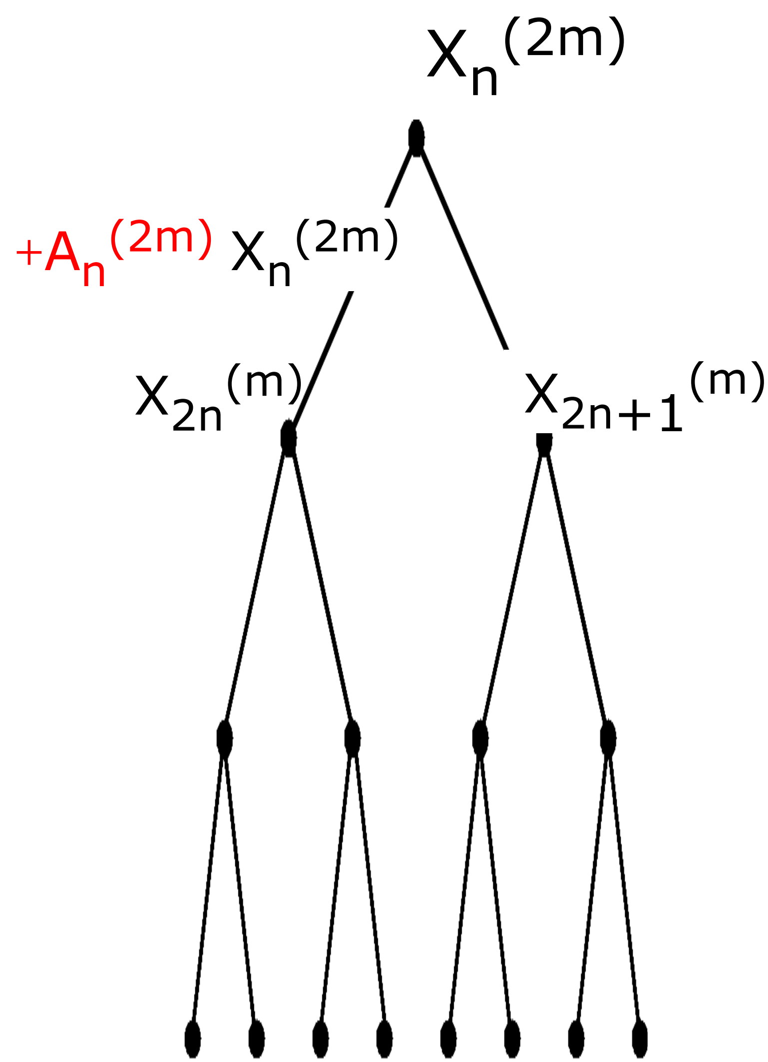
- Introduces dependence X,W
- Model:

  $W_n^{(m)} = A_n^{(m)} \cdot X_n^{(m)}$

  with independent $|A_n^{(m)}|<1$

- Conservation:

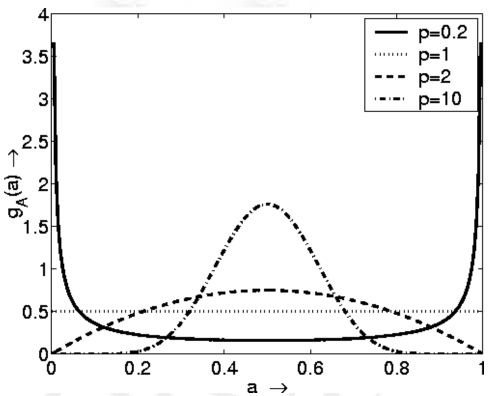
$$X_{2n}^{(m)} = X_n^{(2m)} \cdot (1 + A_n^{(2m)})/2$$
$$X_{2n+1}^{(m)} = X_n^{(2m)} \cdot (1 - A_n^{(2m)})/2$$



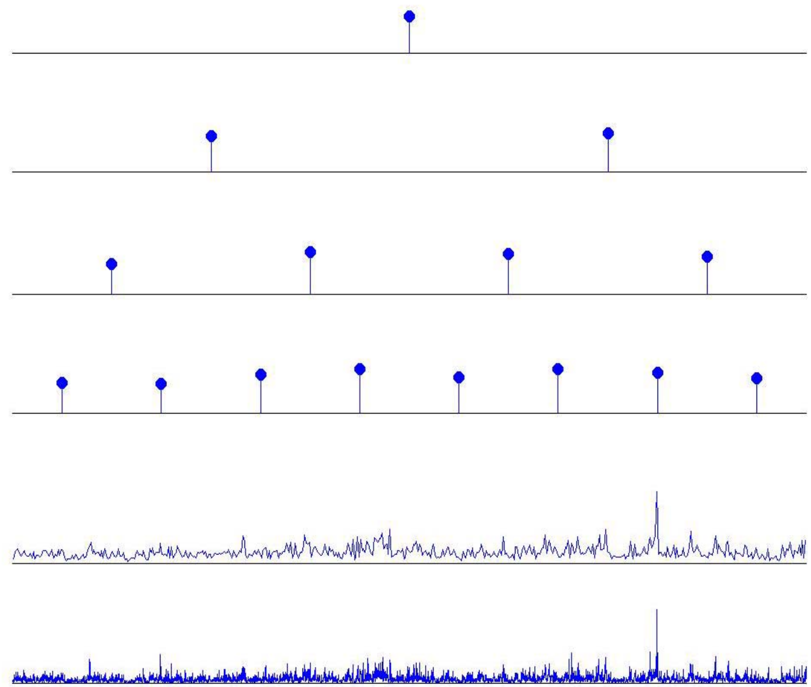
Multiplicative Innovation

Multiplicative Cascade-Model

Multiplicative Innovations



$(1 \pm A_n^{(m)})/2 \sim \mathcal{Beta}\,(\sigma_m)$



positive, spiky (log-normal). LRD

Control variance σ_m to

- Match variance of trace (model fitting)...or...
- match variance progression of LRD with H:

$$\mathrm{Var}X^{(m)} = \mathrm{Var}X^{(2m)} \cdot \frac{1}{4} \cdot \mathrm{Var}(A^{(2m)}) \sim m^{2H} \quad \Rightarrow \mathrm{Var}(A^{(2m)}) = 2^{-2H+2}$$

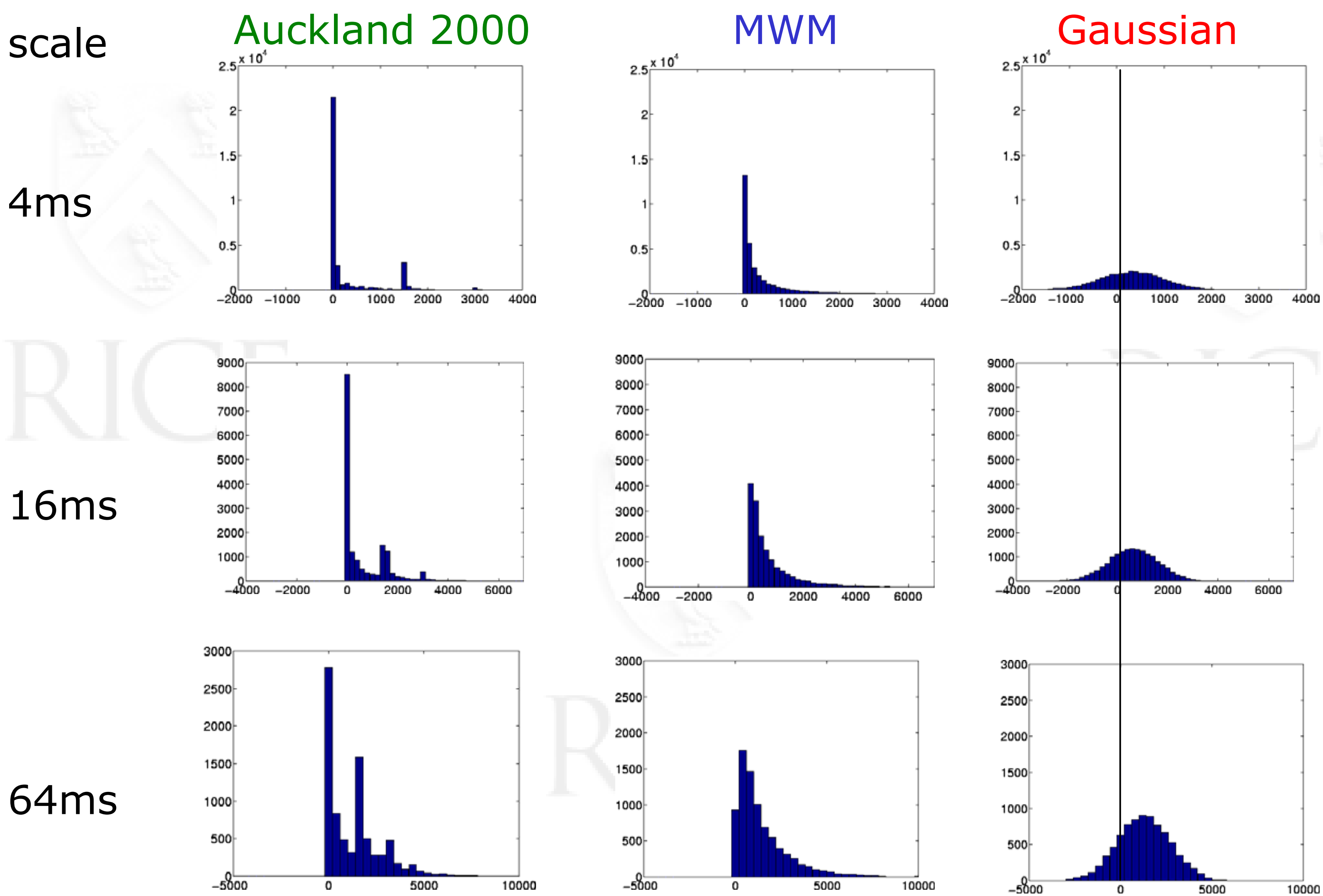
Network relevance

Simulation

Performance (Queuing)

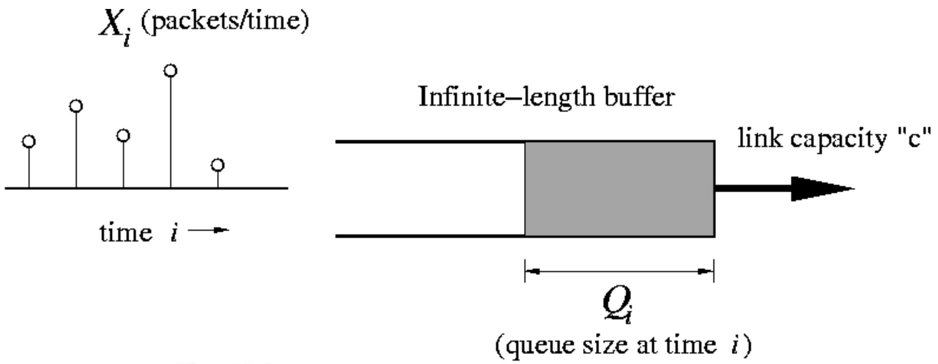
Inference (bandwidth estimation) → later

Multiscale Marginals



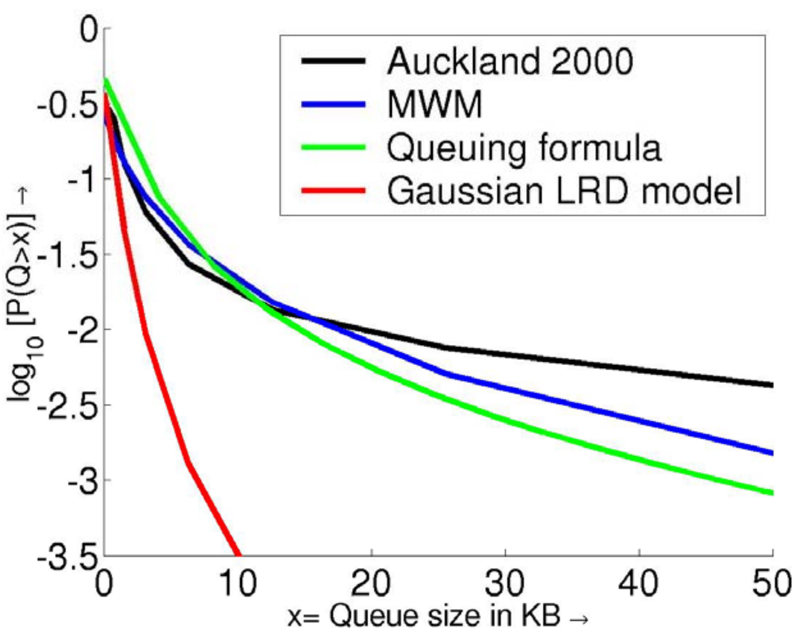
Equal variance on all scales

Queuing analysis



Q-tail: P[Q>b]

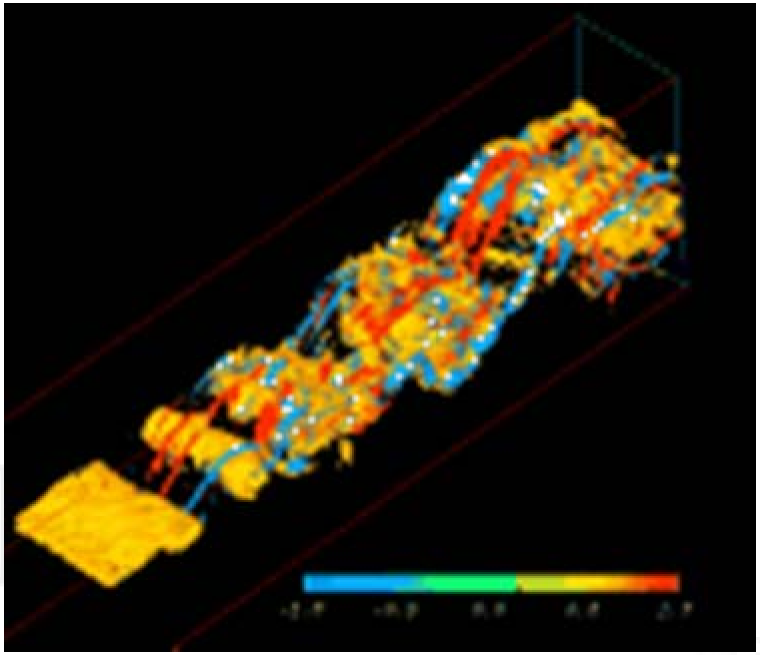
- Tree structure allows for analytical queuing formula
- Multiplicative model superior to additive
- Importance of multiscale marginal distributions

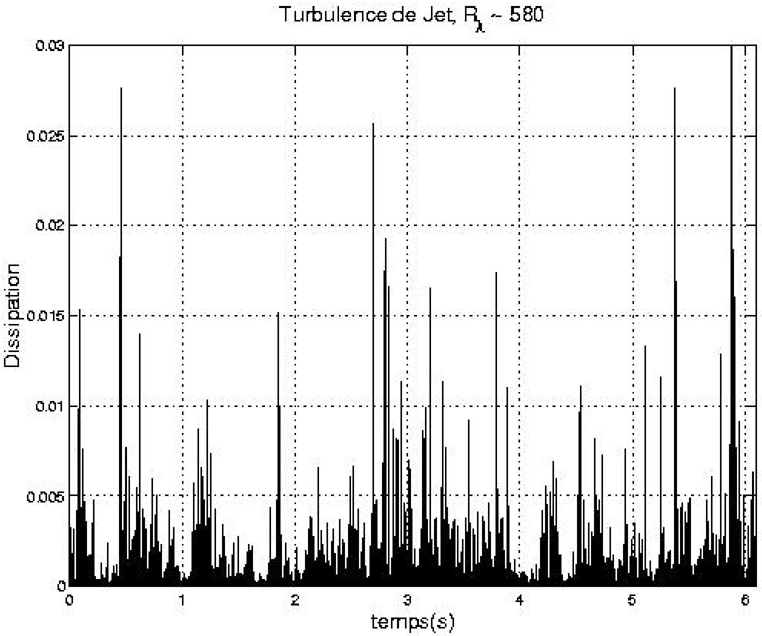


Binomial Cascade

Multifractal Toy

Why Cascades





Courtesy P. Chainais

- Turbulence:

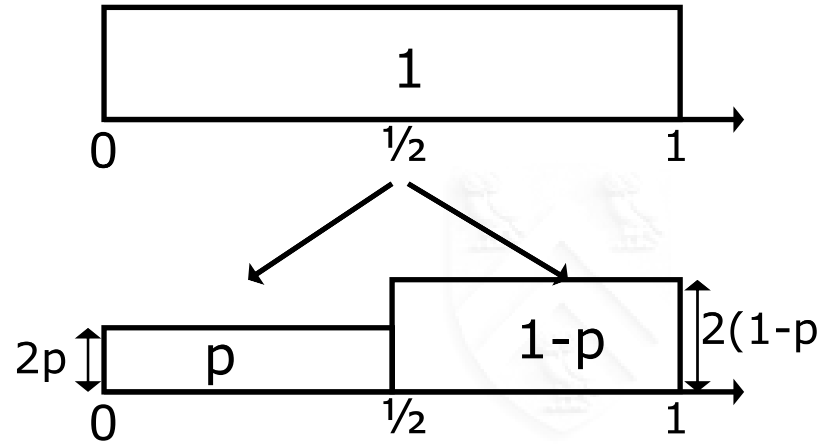
Kolmogorov 41: $\mathbb{E}[|v(t+\delta) - v(t)|^q] \simeq \delta^{q/3} \Rightarrow$ fBm $H = 1/3$

Kolmogorov 62: $\mathbb{E}[|v(t+\delta) - v(t)|^q] \simeq \delta^{\tau(q)}$?????

- Datatraffic: Cascades provide better match

The Toy: Binomial Cascade

- Start with unit mass
- Redistribute uniformly
  portion p <½ to the left
  portion 1-p to the right



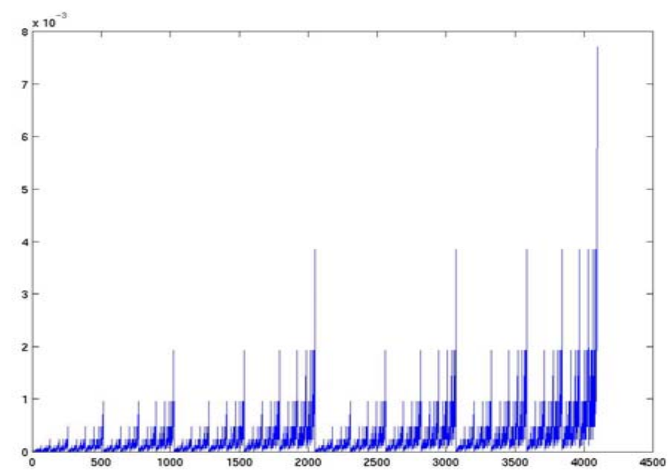
Mass re-distribution

- Iterate

$$t = \sum_{k=1}^{\infty} \epsilon_k / 2^k \quad \text{with } \epsilon_k = 0, 1$$

$$I(\epsilon_1 \ldots \epsilon_n) := [t_n, +1/2^n)) \quad t_n := \sum_{k=1}^{n} \epsilon_k / 2^k$$

$$l_n(t) := \#\{k \leq n : \epsilon_k = 1\} = \sum_{k=1}^{n} \epsilon_k$$



Increment: $X_{2^n t_n}^{(-n)} = p^{n - l_n(t)} (1-p)^{l_n(t)}$

Multifractal Spectrum

- Oscillate ~ $|t|^{\alpha}$ → local strength α

$$\alpha(t) := \liminf_{n} \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

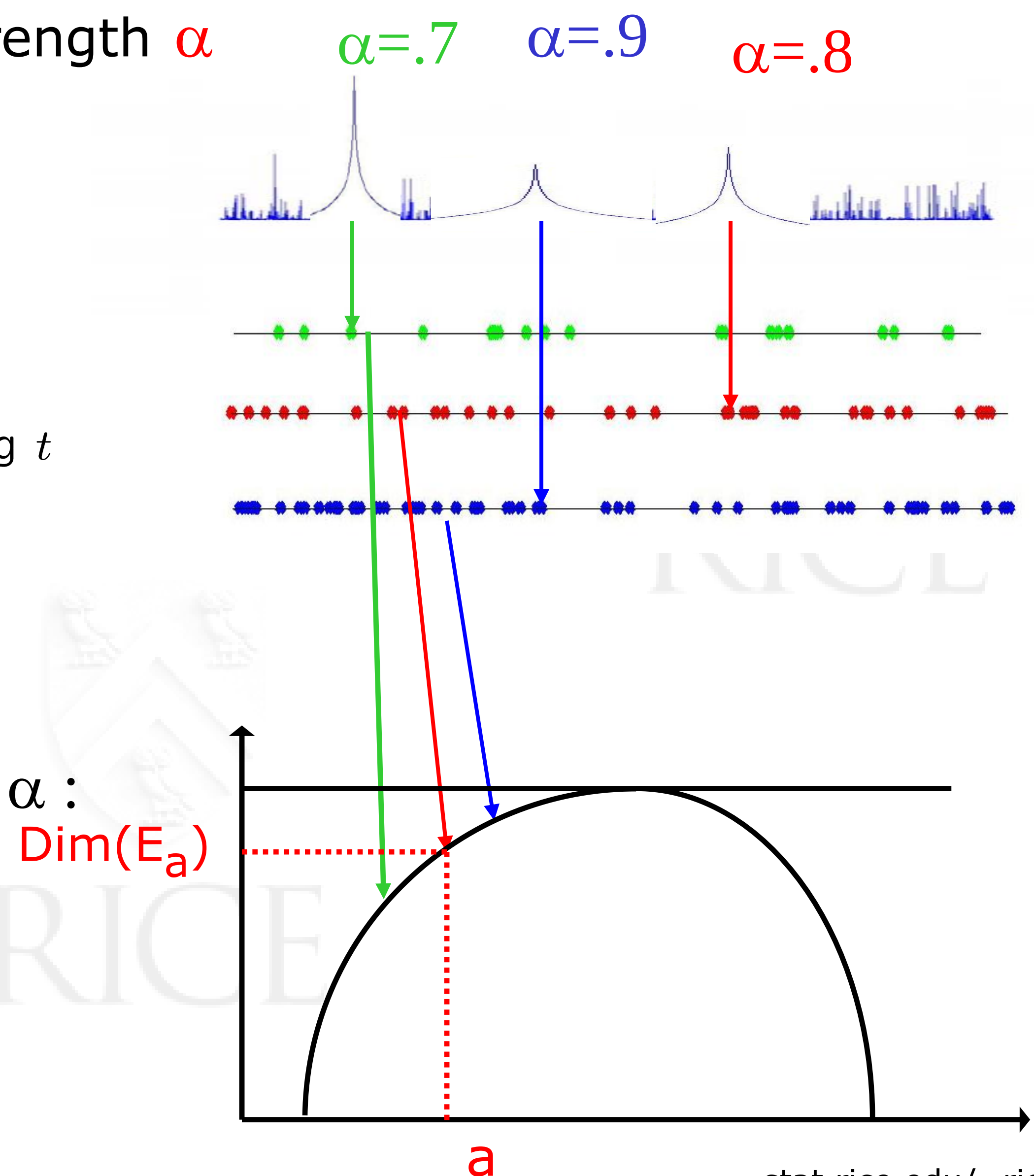
$I_n(t)$: dyadic interval containing t

$\Delta I_n(t)$: oscillation indicator
total increment over I_n,
max increment in I_n,
wavelet coefficients,...

- Collect points t with same α :

$$E_a := \{t : \alpha(t) = a\}$$

- Dim(E_a): Spectrum
→prelevance of α



Binomial

Recall

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \le n : \epsilon_k = 1\}$$

We take dyadic partition:

$$I_n(t) = I(\epsilon_1 \ldots \epsilon_n) := [t_n, t_n + 1/2^n))$$

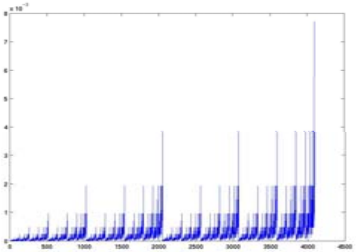
$$\Delta I_n(t) \;=\; X_{2^n t_n}^{(-n)} = p^{l_n(t)}(1-p)^{n-l_n(t)}$$

$$\alpha_n(t) = -\frac{n - l_n(t)}{n}\log_2(p) - \frac{l_n(t)}{n}\log_2(1-p)$$

Range of exponents:

$t = 0$: $l_n = 0$, $\alpha_n \to -\log_2(p)$ > 1: Smooth

$t = 1$: $l_n = n$, $\alpha_n \to -\log_2(1-p) < 1$: Bursty



"Typical" exponents

Recall

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \leq n : \epsilon_k = 1\}$$

t=0, t=1 seem "atypical".

Intuition: for a "typical" t:

$$l_n(t) \simeq n/2$$

Rigorously: Law of Large Numbers

- Binary digits ϵ_k are independent, $P[\epsilon_k=0]= P[\epsilon_k=1]= ½$:
- t is uniformly distributed (i.e., with Lebesgue measure $\mathcal{L}$)
- $$\frac{l_n(t)}{n} = \frac{1}{n}\sum_{k=1}^{n} \epsilon_k \to \mathbb{E}_{\mathcal{L}}[\epsilon] = 1/2$$

- "Typical" exponent:

$$\boxed{\begin{aligned} \alpha_n(t) &= -\frac{n-l_n(t)}{n}\log_2(p) - \frac{l_n(t)}{n}\log_2(1-p) \\ &\to a_0 := -\frac{1}{2}\log_2(p) - \frac{1}{2}\log_2(1-p) \quad > 1 \end{aligned}}$$

A first point on the Spectrum

Conclusion:

- At almost all locations we have a_0, so:

$$\boxed{\dim E_{a_0} = 1}$$

- “Where” or “how many” are the other exponents?

Large Deviations

and the
Multifractal Formalism

Counting via Large Deviations

Recall

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$E_a := \{t : \alpha(t) = a\}$$

- Notation:
  - Number of dyadic intervals with exponent ~ a:

$$N_{n,\delta}(a) := \#\{(\epsilon_1 \dots \epsilon_n) : a - \delta \le \alpha_n(\epsilon_1 \dots \epsilon_n) < a + \delta\}.$$

  - Partition sum: a microscope inspired by LDP

$$S_n(q) := \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q = \sum_{\epsilon_1 \dots \epsilon_n} |2^n|^{q\alpha_n(\epsilon_1 \dots \epsilon_n)}.$$

  - Assume powerlaws:

$$N_{n,\delta}(a) \sim 2^{nf(a)} \qquad S_n(q) \sim 2^{-n\tau(q)}$$

  - Typically (LDP) $\boxed{f(a) = \inf_q (qa - \tau(q))}$

LDP and the Legendre transform

Recall

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$E_a := \{t : \alpha(t) = a\}$$

- Finding the dominating terms in S(q):

$$\begin{aligned}
2^{-n\tau(q)} \sim S_n(q) &= \sum_{(\epsilon_1 \dots \epsilon_n)} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \\
&= \sum_{l=1}^{m} \sum_{\alpha_n(\epsilon_1 \dots \epsilon_n) \in [l\delta - \delta/2, l\delta + \delta/2]} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \\
&\sim \sum_{l=1}^{m} N_{n,\delta/2}(l\delta) \cdot 2^{-nql\delta} \\
&\sim \sum_{l=1}^{m} 2^{-n(ql\delta - f(l\delta))} \\
&\sim 2^{-n(\inf_a (qa - f(a)))}
\end{aligned}$$

- …shows that τ and f are Legendre pairs

$$\tau(q) = \inf_a (qa - f(a)) \qquad f(a) = \inf_q (qa - \tau(q))$$

Legendre spectrum

- Thm: provided $\alpha_n(t)$ are bounded we have

$$\boxed{f(a) = \tau^*(a) \qquad \text{for } a = \tau'(q).}$$

- …in other words

$$\#\{(\epsilon_1 \ldots \epsilon_n) : a - \delta \leq \alpha_n(\epsilon_1 \ldots \epsilon_n) < a + \delta\}$$
$$\sim 2^{n \inf_q (qa - \tau(q))}$$

- …and the multifractal spectrum is the Legendre transform of the partition scaling exponent

Legendre transform 101

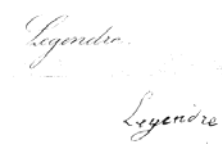
- Elementary calculus:

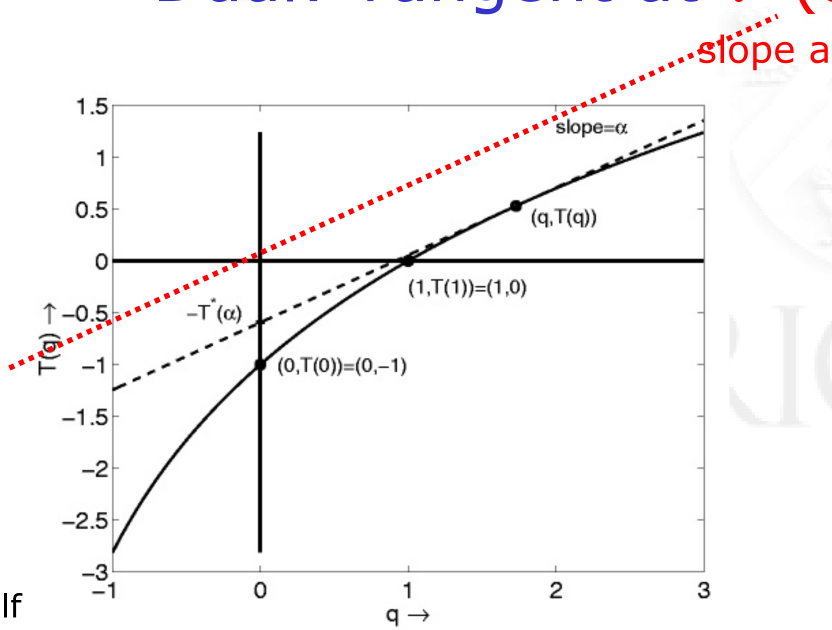
$$\tau^*(a) := \inf_q (qa - \tau(q)) = \overline{q}a - \tau(\overline{q})$$

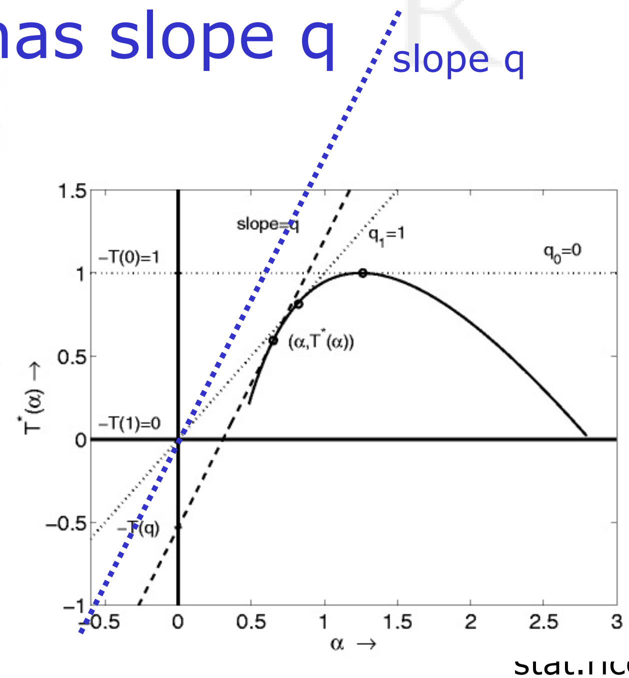
where $\overline{q}$ is defined by $a = \tau'(\overline{q})$

  - Draw tangent of slope a to τ(q).
  - The intersection with y-axis yields -τ*(a)
  - Dual: Tangent at τ*(a) has slope q

Legendre







Binomial Spectrum

continued

Multifractal analysis of the Binomial

$$\begin{aligned} S_n(q) &= \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \\ &= \sum_{\epsilon_1 \ldots \epsilon_n} [p^{n-l_n(\epsilon_1 \ldots \epsilon_n)}(1-p)^{l_n(\epsilon_1 \ldots \epsilon_n)}]^q \\ &= \sum_{l=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [p^{n-l}(1-p)^l]^q \\ &= [p^q + (1-p)^q]^n. \end{aligned}$$

- Partition function

$$\tau(q) = \lim_{n\to\infty} -\frac{1}{n} \log_2 S_n(q) = -\log_2[p^q + (1-p)^q]$$

- Via Legendre: Most often we see exponent a_0 such that $f(a_0)$ is maximal. This happens where the tangent is horizontal, thus where q=0. So, as before:

$$a_0 = \tau'(0) = -\frac{1}{2}\log_2(p) - \frac{1}{2}\log_2(1-p) \quad > 1$$

Insight from Large Deviations

- From steepest ascent:

$$
\begin{aligned}
S_n(q) &= \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \simeq 2^{-n(\inf_a(qa - f(a))} \\
&= 2^{-n(q\bar{a} - f(\bar{a}))} \simeq \sum_{\alpha_n(\epsilon_1 \ldots \epsilon_n) \simeq a} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q
\end{aligned}
$$

  - Dominant terms in $S_n(q)$, for fixed q, are the ones with

$$
\alpha_n(\epsilon_1 \ldots \epsilon_n) = \frac{\log \Delta I_n}{\log |I_n|} \simeq \bar{a} = \tau'(q)
$$

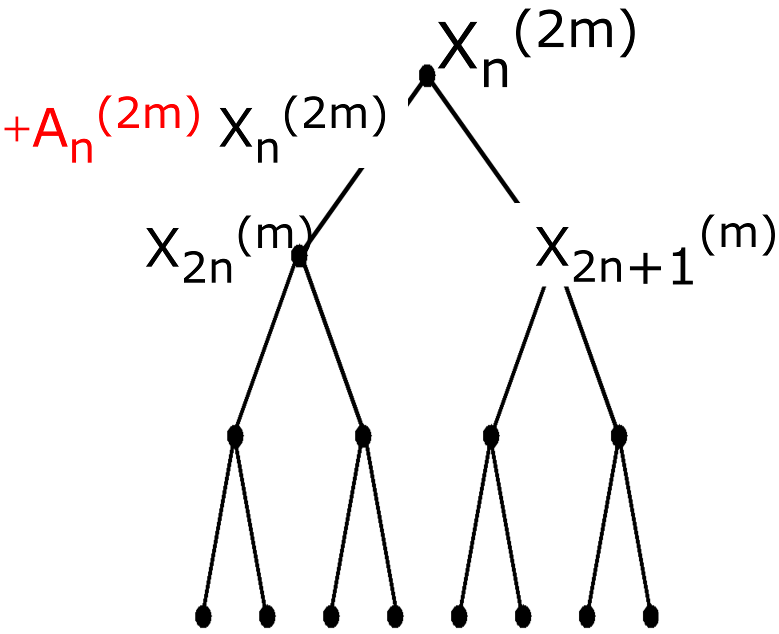
    - …and vice versa: these terms contribute such that

$$
S_n(q) \simeq 2^{-n\tau(q)} = (p^q + (1-p)^q)^n
$$

    - For the Binomial these correspond to choosing digits in the ratio p^q to $(1-p)^q$

Spectrum of the MWM

Multifractal Wavelet Model



Choose *independent* r.v. $A_{(j)}$ *symmetrically* distributed in $[-1,1]$.

Define recursively

$$W_{j,k} = A_{(j)} \cdot U_{j,k}$$

Resulting *stationary* (1st order) series

$$X_k \stackrel{d}{=} U_{J_0,0} \cdot \prod_{j=J_1}^{J_0} (1 + A_{(j)})/2$$

$$X_{2n}^{(m)} = X_n^{(2m)} \cdot (1 + A_n^{(2m)})/2$$

$$X_{2n+1}^{(m)} = X_n^{(2m)} \cdot (1 - A_n^{(2m)})/2$$

Multifractal analysis of the MWM

- Partition function

$$\frac{-1}{n}\log_2 \underbrace{\sum_{i=1}^{n}\sum_{k_i=0,1}}_{\text{sum over the } 2^n \text{ dyadic points of order } n} \mathbb{E}\left(\prod_{i=1}^{n}(1+(-1)^{k_i'}A_{(-i,k_i)})\right)^{q}$$

Binomial formula

$$\doteq \frac{-1}{n}\log_2 \prod_{i=1}^{n}\left(\mathbb{E}(1+A_{(-i)})^q+\mathbb{E}(1-A_{(-i)})^q\right)$$

Symmetry of $A_{(-i)}$

$$\doteq -1-\frac{1}{n}\sum_{i=1}^{n}\log_2 \mathbb{E}[(1+A_{(-i)})^q]$$

$$\to -1-\log_2 \mathbb{E}[(1+A)^q] \qquad \text{provided } A_{(j)} \stackrel{distr}{\to} A.$$

- Special case of Beta-variables A

$A_{(j)} \simeq \beta(p_{(j)},p_{(j)})$ with $p_{(j)} \to p$ as $j \to -\infty$

$$\tau(q) = -1-\log_2 \frac{\Gamma(p+q)\Gamma(2p)}{\Gamma(2p+q)\Gamma(p)}.$$

Spectrum of self-similar processes

Mono-fractals

MFA of Self-similar processes

Assume Y is H-sssi with increments

$$X_k^{(n)} = Y(k2^{-n}) - Y((k-1)2^{-n})$$

$$\mathbb{E} \sum_{k=1}^{2^n} |X_k^{(n)}|^q = 2^n \mathbb{E}|X_1^{(n)}|^q = 2^{n-nqH} \mathbb{E}|X_1^{(1)}|^q,$$

Assume $\mathbb{E}|X_1^1|^q < \infty$ for $q_{\text{bot}} < q < q_{\text{top}}$.

fBm: $q_{\text{bot}} = -1$, $q_{\text{top}} = \infty$,

α-stable process: $q_{\text{bot}} = -1$, $q_{\text{top}} = \alpha$.

Then

$$\tau(q) = \begin{cases} qH - 1 & \text{for } q_{\text{bot}} < q < q_{\text{top}}, \\ -\infty & \text{else.} \end{cases}$$

Linear Spectrum!

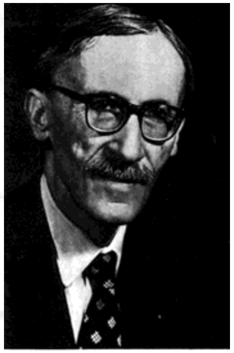
$$f(a) = \begin{cases} 1 + q_{\text{top}}(\alpha - H) & \text{for } \alpha < H \\ 1 + q_{\text{bot}}(\alpha - H) & \text{for } \alpha \geq H. \end{cases}$$

Wavelets

A powerful multiscale tool

History of wavelets

- Fourier series (1807)
- Levy (1930): Haar basis superior to Fourier for Brownian motion
- Weiss-Coifman (‘60-’80):
  - decompose functions into atoms
- Grossman-Morlet ’80: defined wavelets
- Mallat ’85: pyramidal algorithm, o.n. basis
- →Meyer: continuously diff wavelets
- →Daubechies: compactly supported wavelets



PAUL LÉVY 1886-1971

Ortho-normal Wavelets

- Multi-resolution analysis (Mallat, Daubechies):
  - There are compactly supported ψ and ϕ s.t.

$$\psi_{j,k}(t) \;:=\; 2^{-j/2}\,\psi\Big(2^{-j}t-k\Big)$$

$$\phi_{j,k}(t) \;:=\; 2^{-j/2}\,\phi\Big(2^{-j}t-k\Big)$$

  form orthonormal bases of $\mathcal{L}^2$.

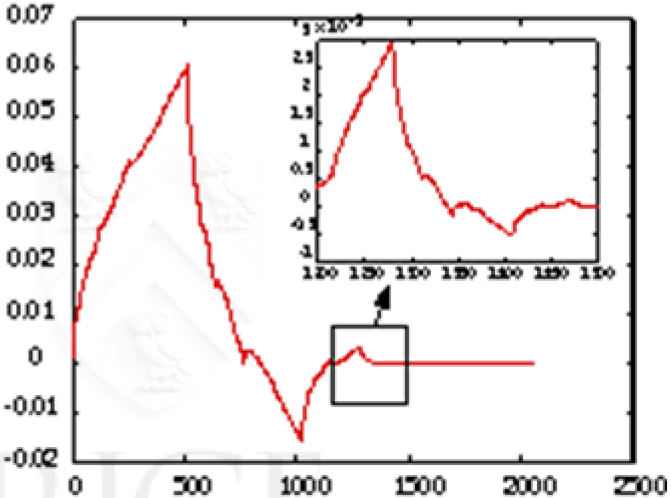
  - For X supported on $[0,2^{J_0}]$

$$X(t) \;=\; \sum_k U_{J_0,k}\,\phi_{J_0,k}(t) + \sum_{j=-\infty}^{J_0}\sum_k W_{j,k}\,\psi_{j,k}(t),$$

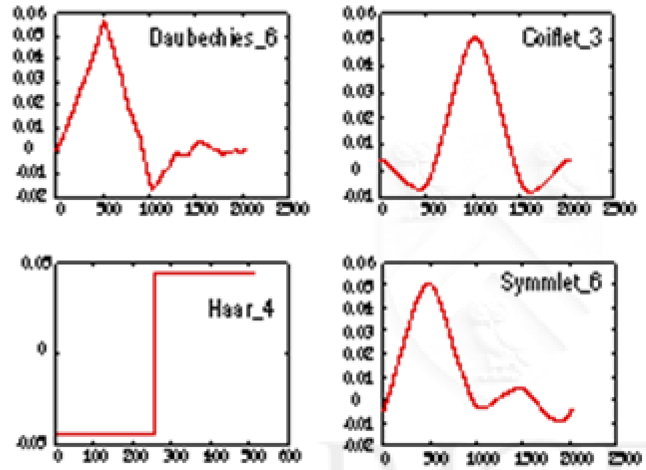
  with

$$W_{j,k} := \int X(t)\,\psi^*_{j,k}(t)\,dt \quad \text{and} \quad U_{j,k} := \int X(t)\,\phi^*_{j,k}(t)\,dt.$$

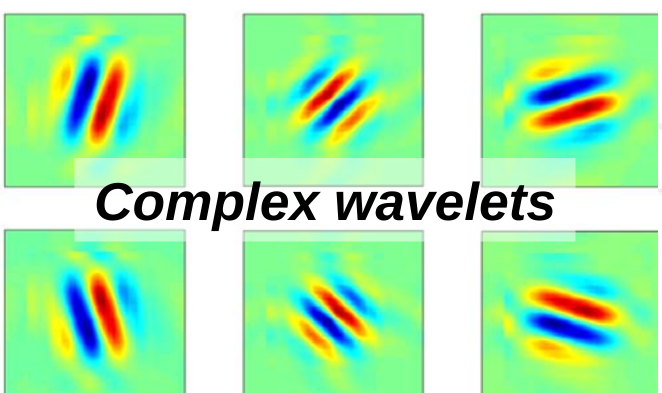
Wavelets: what they look like



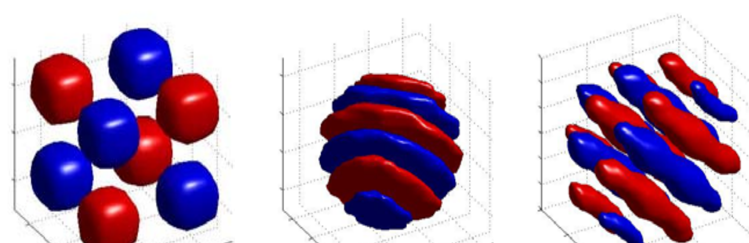
Daubechies 4 Mother wavelet



Daubechies 6 Coiflet 3
Haar 4 Symmlet 6



Complex wavelets



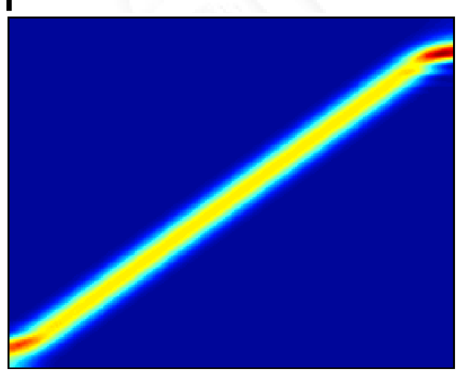
Multidimensional wavelets

Continuous wavelets

- Continuous rescaling of mother wavelet
- Continuous (redundant) set of coefficients
- Often used: Mexican hat $(\exp(-x^2))''$

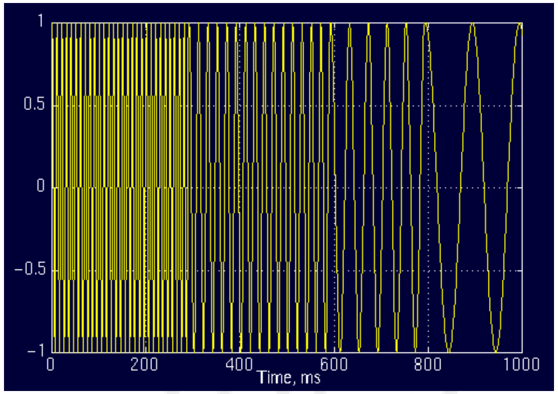
$$T(a,t) = \frac{1}{a}\int X(s)\psi\left(\frac{s-t}{a}\right)ds$$

- Form of a convolution → Fourier, Parseval



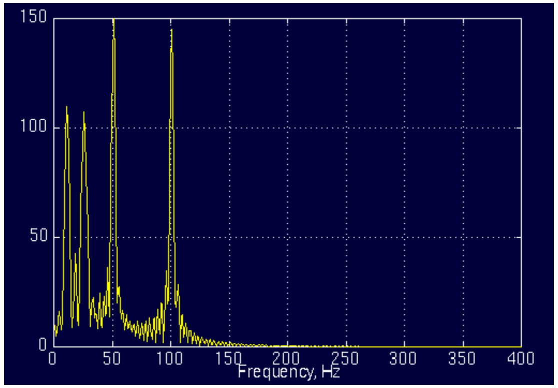
Wavelet vs Fourier

- Fourier
  - timing information is hidden in the phase
  - sin(t) and cos(t) are not localized in time



Signal

- Power spectral density
  - Identifies frequency content only, but not their location
  - Relation to Auto-correlation

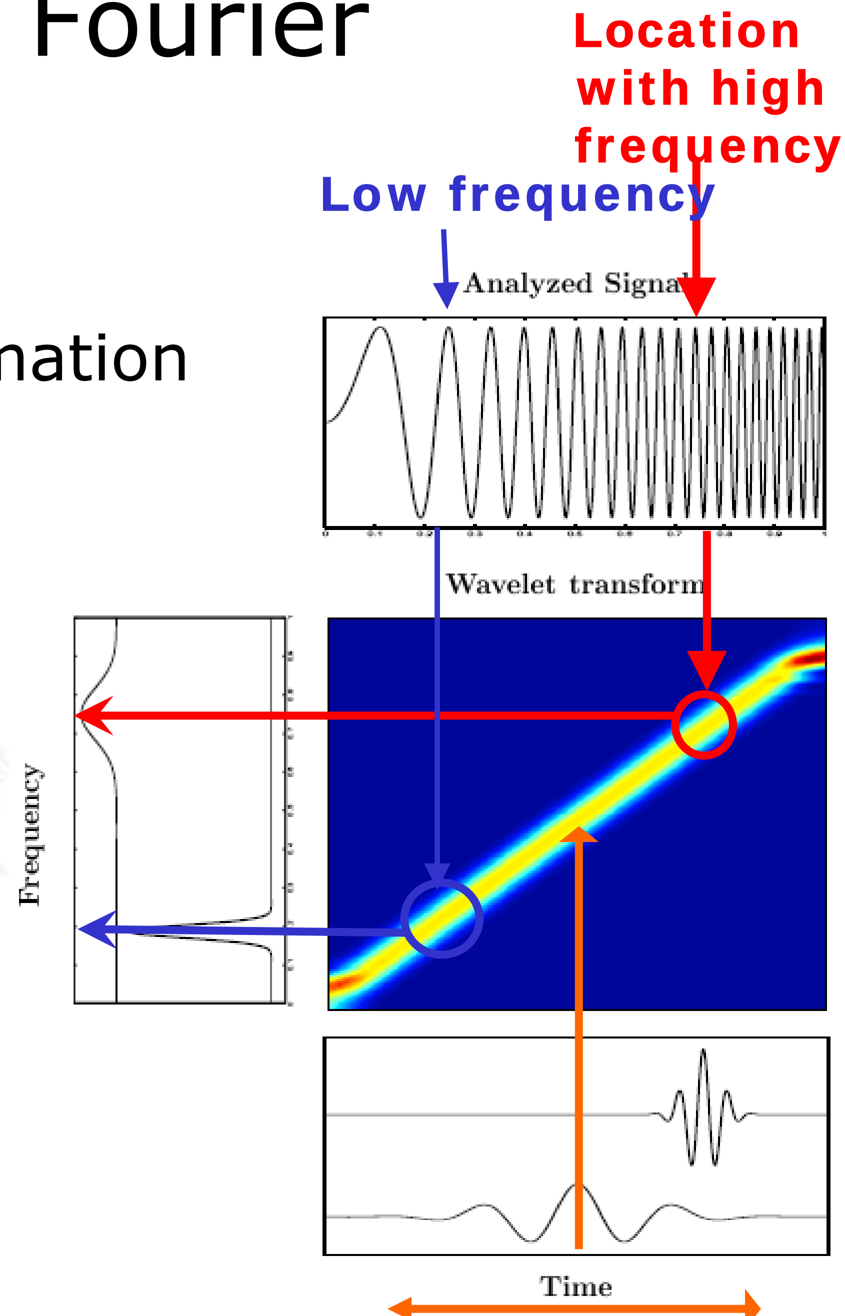


Fast Fourier Transform

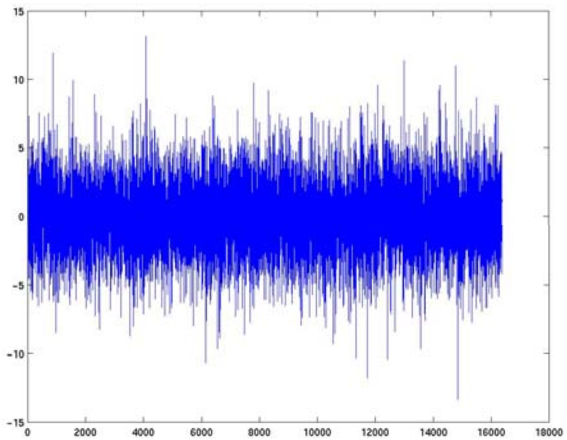
Wavelet vs Fourier

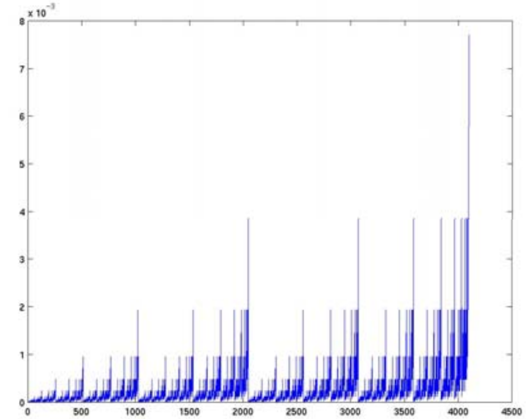
- Power spectrum
  - provides no timing information
  - sin(t) and cos(t) are not localized in time

- Wavelets are localized
  - both in time
  - and in frequency



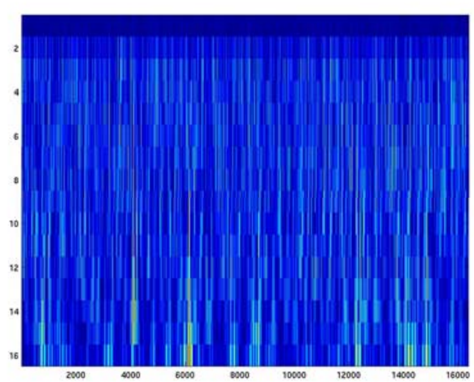
Toy examples

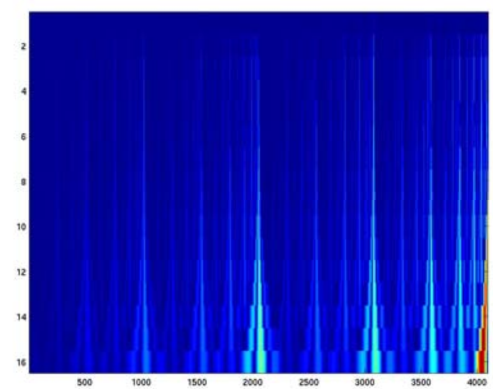




White noise

Cascade



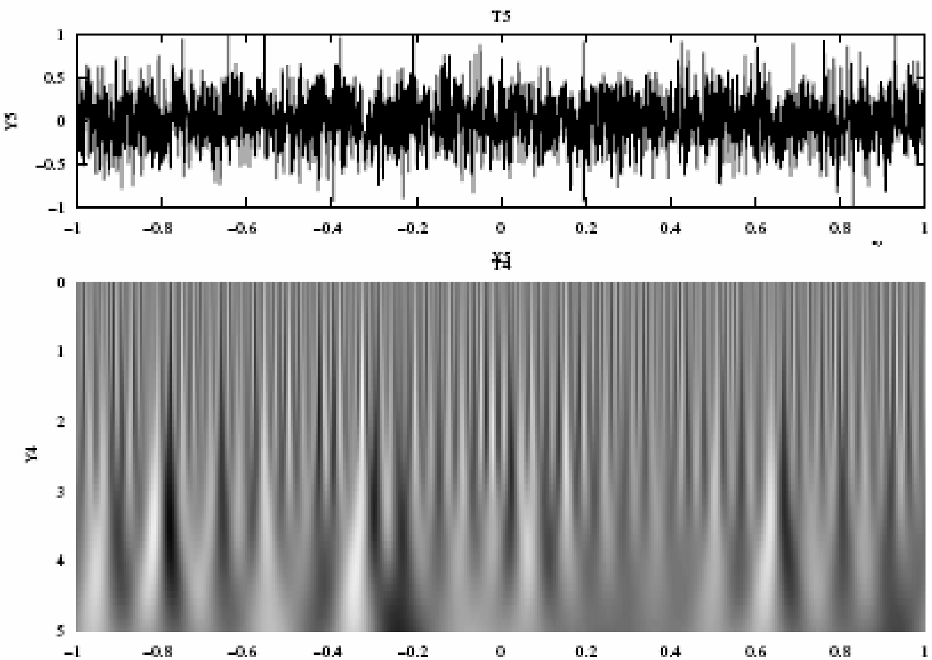


Wavelet trafo indicates:
Mono-fractal

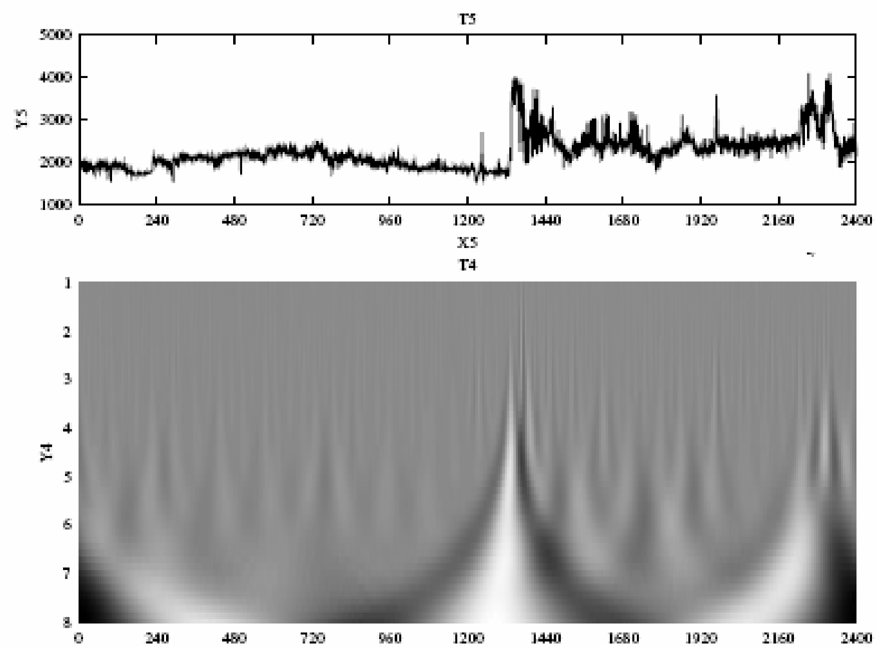
Wavelet trafo indicates:
Multi-fractal

Continuous wavelet at work

- Wavelets are an excellent tool to identify local frequency content



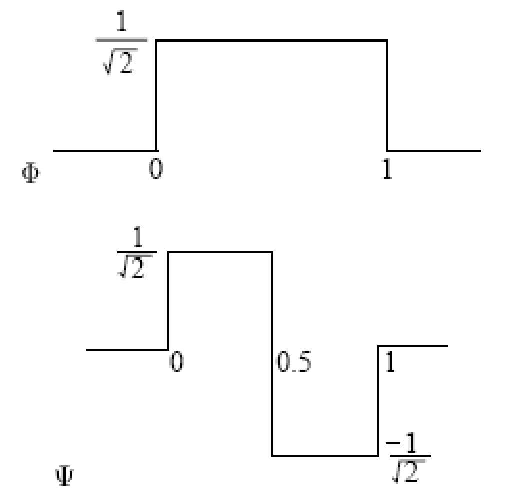
Fractional Gaussian Noise



Geological Well data

Haar wavelet

- Haar wavelet (See plot)



Haar scaling Φ and wavelet Ψ functions.

- Clearly

$$\psi_{j,k}(t) \;:=\; 2^{-j/2}\,\psi\big(2^{-j}t-k\big)$$

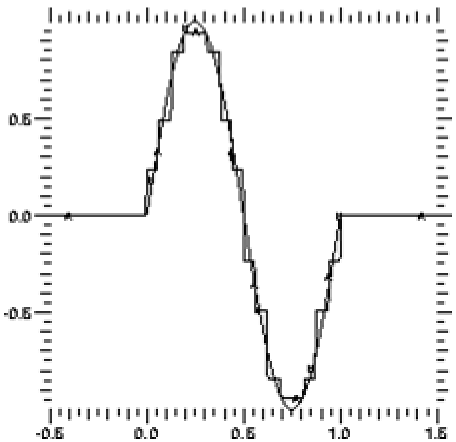
$$\phi_{j,k}(t) \;:=\; 2^{-j/2}\,\phi\big(2^{-j}t-k\big)$$

form orthonormal bases of $\mathcal{L}^2$.

- Coefficients
  - $$W_{j,k} = 2^{-j/2}\left[\int_{2^jk}^{2^j(k+\frac{1}{2})} f(x)dx - \int_{2^j(k+\frac{1}{2})}^{2^j(k+1)} f(x)dx\right]$$
  - For $Z(t)=\int_0^t f dx,\ X_k^{(m)} = Z(km)-Z(km-m)$

$$W_{j+1,k} = X_{2k}^{(2^j)} - X_{2k+1}^{(2^j)}$$

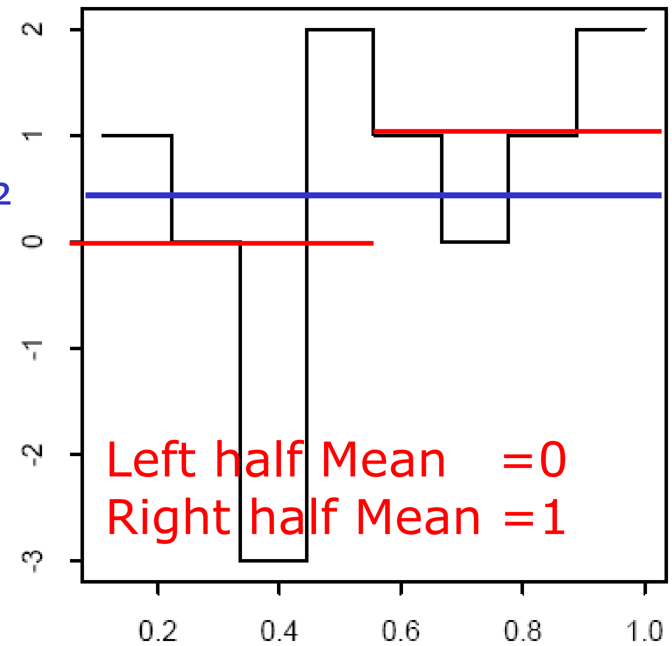
Haar wavelet at work



Haar approximation of sin(x)

Haar approximation of some f(x)

Overall Mean= ½



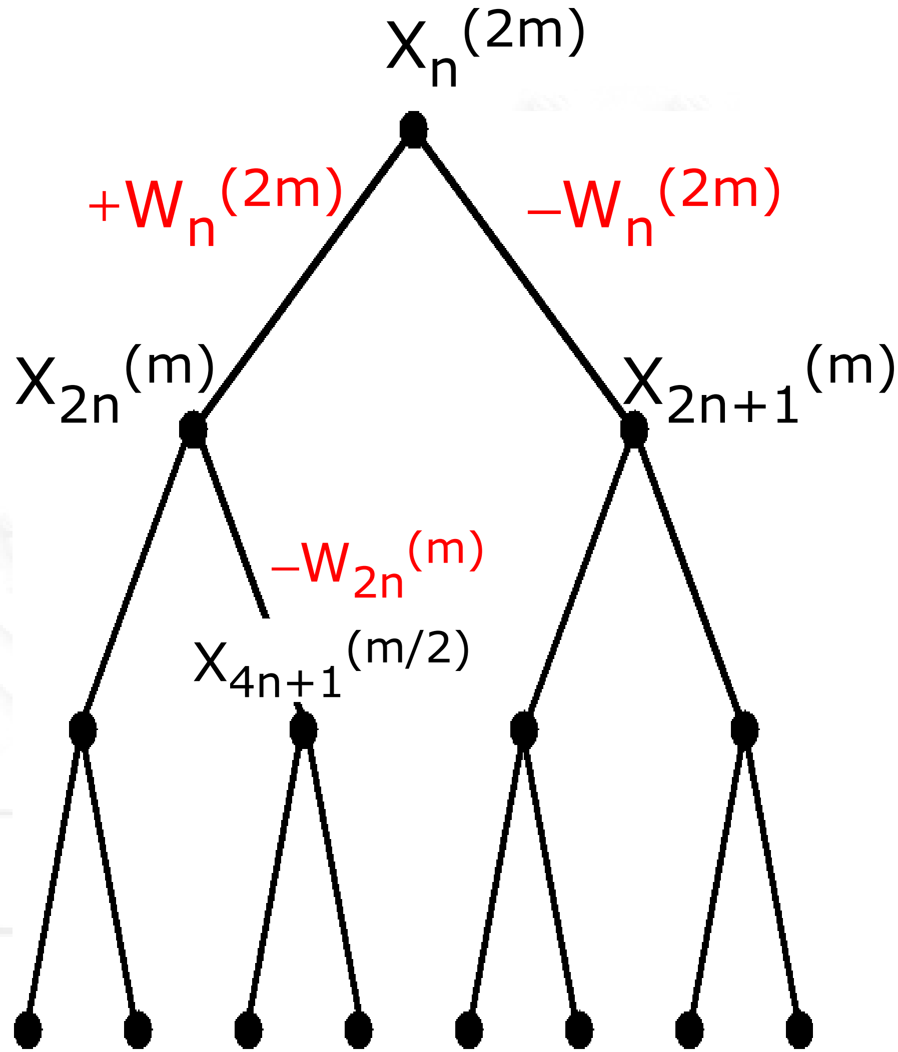
$$f = \frac{1}{2}\phi - \frac{1}{2}\psi_{00} + \frac{1}{2\sqrt{2}}\psi_{10} - \frac{1}{2\sqrt{2}}\psi_{11} + \frac{1}{4}\psi_{20} - \frac{5}{4}\psi_{21} + \frac{1}{4}\psi_{22} - \frac{1}{4}\psi_{23}$$

Additive Tree is Haar Model

Synthesis:

- Start at root
- Flow down the tree
- Additive, independent innovations $W_n^{(m)}$
- these are essentially the Haar wavelet coefficients

- Idea: use any wavelet
  - The better the frequency response, the better the spectral approximation to fBm



LRD vs. Large Deviations

Large vs Small scales

Doubly stochastic modeling

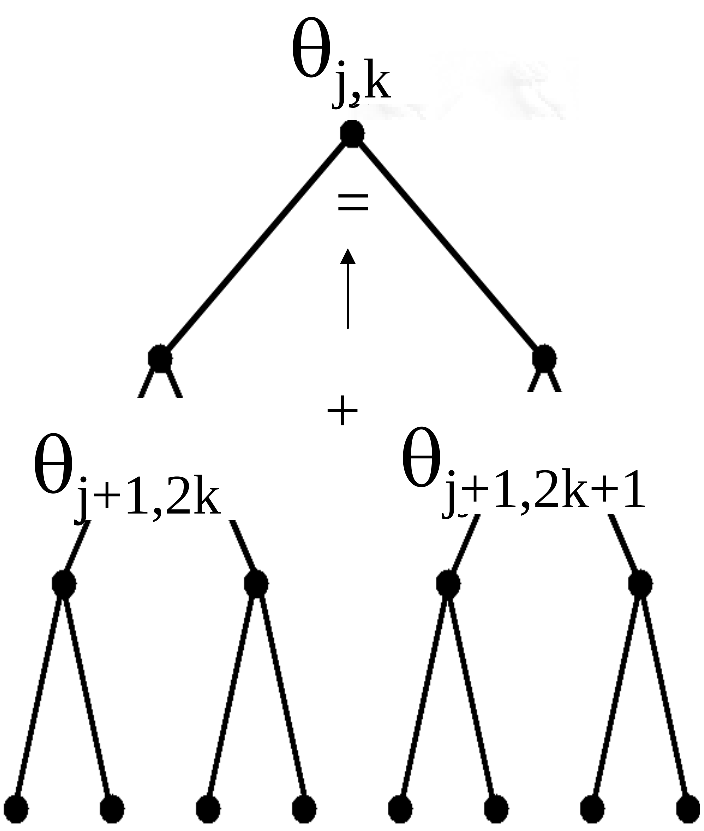
Setting:

- Background process $\theta_{j,k}$
- Given $\theta_{j,k}$ the observed multiscale loads $U_{j,k}$ are *independent* of *mean* $\theta_{j,k}$

- $E[U_{j,k} \mid \theta] = \theta_{j,k}$

- $\theta_{j,k}$ fill a multiscale tree

$$\theta_{j+1,2k} + \theta_{j+1,2k+1} = \theta_{j,k}$$



Gaussian versus Poisson

- Gaussian

$U_{j,k} = \mathcal{N}(\theta_{j,k}, \sigma^2_{j,k})$

- Poisson

$U_{j,k} = \mathcal{P}(\theta_{j,k})$

Iteration scheme for synthesis

$U_{j+1,2k} \mid U_{j,k} =$

$\mathcal{N}(U_{j,k}/2 + (\theta_{j+1,2k} - \theta_{j+1,2k+1})/2 \,,\, \sigma^2_{j,k}/2)$

additive innovation

$\mathcal{B}inom(U_{j,k} \,,\, \theta_{j+1,2k} / \theta_{j,k})$

multiplicative innovation

Scaling from a modeling perspective

- ON-OFF limits and LRD
  - User driven: heavy tail file sizes
  - Additive, Gaussian
  - Large scales
- Cascades and multifractal scaling
  - Network driven: heterogeneity of RTT
  - Multiplicative, log-Normal
  - Small scales