A non-parametric wavelet-based estimator of tails and Internet Traffic

Rolf Riedi

with
Paulo Goncalves, INRIA Rhone-Alpes

Hailuoto, June 2005

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Diverging Moments

Diverging moments: $\mathbb{E}|X|^q = \infty$

bear on…

- Estimation of tails: $P[|X| > x] \sim x^{-\alpha}$
- Estimators per se:

$$(X_1^2 + ...X_n^2)/n$$

  - Bias
  - Asymptotic normality

Theory

Tails

- Let λ>0.

$$\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda}$$

$$\Leftrightarrow$$

$$\boxed{P[|X| > 1/u] \overset{u \rightarrow 0}{=} O(|u|^r) \quad \text{for all } r < \lambda}$$

- “Moments and Tails go together”

Characteristic function 101

- Characteristic function:

$$\phi(u) = \mathbb{E}[\exp(iuX)]$$

- Moments 101: If E|X|^n is exists then

$$\phi^{(n)}(0) = i^n \mathbb{E}[X^n]$$

- Vice versa: If ϕ has 2p derivates then E|X|^{2p} exists

Characteristic function 102

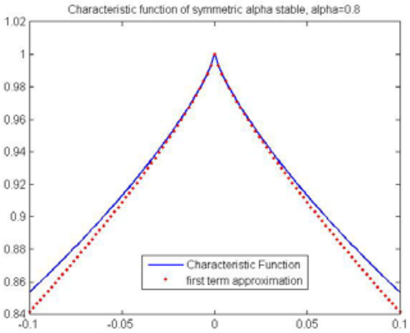
- Moments 102: (Tauberian Thm) For 0<λ<2

$$\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda} \qquad \Leftrightarrow$$

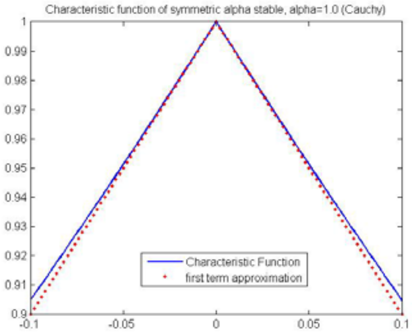
$$\boxed{\mathcal{R}e\ \phi(u) - 1 \overset{u\to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda}$$

- Example: symmetric stable laws (moments up to α)

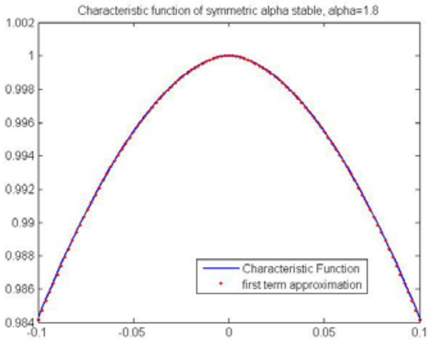
$$\phi(u) = \mathcal{R}e\ \phi(u) \ = \ \underline{\exp(-|u|^\alpha)} \simeq \underline{1 - |u|^\alpha} + \ldots \qquad (|u| \to 0)$$



$\alpha = 0.8$



$\alpha = 1$: Cauchy



$\alpha = 1.8$

Extension to orders > 2

- Kawata ('72) / Lukacs ('83) / Ramachandran ('69):
  - Let $2p<\lambda\leq 2p+2$ with integer p.
  - If $\mathbb{E}|X|^\lambda < \infty$ then $\mathcal{R}e\ \phi(u) - \sum_{k=1}^{p} \frac{(-1)^k}{(2k)!}\mathbb{E}[X^{2k}]u^{2k} = O(|u|^\lambda)$
  - Vice versa: If $\mathcal{R}e\ \phi(u) - \sum_{k=1}^{p} a_{2k}u^{2k} = O(|u|^\lambda)$ then $\mathbb{E}|X|^r < \infty$ for all $r < \lambda \ldots$
  - (upon inspection of proof): provided the $a_{2k} = \frac{(-1)^k}{(2k)!}\mathbb{E}[X^{2k}]$ exist.
- Summary: $\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda}$ $\Longleftrightarrow$

$$\boxed{\mathcal{R}e\ \phi(u) - \sum_{k=1}^{p} \frac{(-1)^k}{(2k)!}\mathbb{E}[X^{2k}]u^{2k} \overset{u\to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda}$$

Estimating the Regularity of ϕ

- Motivation:
  - exact regularity of ϕ at zero provides the cutoff value for finite moments
- Microscope for regularity: Wavelet transform T

$$T(a,b) = <\mathcal{R}e\ \phi, \underline{\psi_{a,b}}> = \int \mathcal{R}e\ \phi(s) \cdot \underline{\frac{1}{a}\psi\left(\frac{s-b}{a}\right)}\, ds$$

- Simplified regularity theorem: Assume
  - Wavelet regularity N>λ: $\int t^k \psi(t)dt = 0 \quad \text{for } 0 \le k < N$
  - Hoelder polynomial Pϕ of degree ≤N
  - *Transform T(a,t) is maximal at 0*
  - Then

$$\boxed{\begin{array}{lrl} & \mathcal{R}e\ \phi(u) - P_\phi(u) \overset{u\to 0}{=} O(|u|^r) & \text{for all } r < \lambda \\ \Leftrightarrow & T(a,0) \overset{a\to 0}{=} O(|a|^r) & \text{for all } r < \lambda \end{array}}$$

Proof of simplified regularity theorem:

- If (1) $\phi(u) - P_\phi(u) \overset{u\rightarrow 0}{=} O(|u|^r)$

  and if (2) the wavelet ψ is supported on [0,1]
- then $T(a,0) \overset{a\rightarrow 0}{=} O(|a|^r)$

$$\begin{aligned}
|T(a,0)| = \left|< \phi, \psi_{a,0} >\right| \quad &= \left|\frac{1}{a}\int_0^a \phi(s)\,\psi(s/a)\; ds\right| \\
\text{(2)} \quad &= \left|\frac{1}{a}\int_0^a (\phi(s) - P_\phi(s))\,\psi(s/a)\; ds\right| \\
\int t^k\psi(t)dt = 0 \quad \text{for } 0 \le k < N \quad &\le C \cdot \frac{1}{a}\int_0^a |s|^r \; |\psi(s/a)|\, ds \\
\text{(1)} \quad &\le C \cdot a^r \frac{1}{a}\int_0^a |\psi(s/a)|\, ds \\
&\le C \cdot a^r \cdot \int_{\mathbb{R}} |\psi(s)|\, ds
\end{aligned}$$

Wavelet Transform of ϕ

- Fourier transform:

$$\Psi_{a,b}(x) = \frac{1}{a}\int \psi(\frac{s-b}{a})e^{isx}ds = \int \psi(u)e^{i(au+b)x}du = e^{ixb}\Psi(ax)$$

- Parseval:

$$T(a,b) =< \mathcal{R}e\ \phi, \psi_{a,b} >= \mathcal{R}e < F, \Psi_{a,b} >= \mathcal{R}e\,\mathbb{E}[\Psi_{a,b}(X)]$$

- Assume: Fourier Transform Ψ of ψ is real positive.
  - then:

$$|T(a,b)| \le \mathbb{E}[|\Psi_{a,b}(X)|] = \mathbb{E}[|\Psi(aX)|] = |T(a,0)|$$

  - in other words: *T(a,0)* *maximal*

- Ex:

$$\Psi(u) = u^{2n}\exp(-u^2) \ge 0$$

$$\psi(x) = (-1)^n(\tfrac{d}{dx})^{2n}\exp(-x^2)$$

Wavelet Transform of ϕ

- Parseval:

$$T(a,b) = \mathbb{E}[\Psi_{a,b}(X)] = \mathbb{E}[e^{ixb}\Psi(ax)]$$

- Assume: Ψ is real positive then T(a,0) maximal
- Recall equivalent conditions for $0<\lambda<2$:

$$(1) \quad \mathcal{R}e\ \phi(u) - 1 \overset{u\to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda$$

$$(2) \quad T(a,0) \overset{a\to 0}{=} O(|a|^r) \quad \text{for all } r < \lambda$$

- → estimate regularity of Re(ϕ) by the powerlaw

$$\boxed{|T(a,0)| = \mathbb{E}[|\Psi(aX)|] \sim a^{\lambda}}$$

Extension to orders > 2: Differentiability

- Recall Kawata'72 / Lukacs'83 / Ramachandran'69:

$$\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda} \quad \Leftrightarrow$$

$$\boxed{\mathcal{R}e\ \phi(u) - \sum_{k=1}^{p} \frac{(-1)^k}{(2k)!} \mathbb{E}[X^{2k}] u^{2k} \overset{u \to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda}$$

- Wavelets are blind to *any* polynomials, provide no estimate of differentiability:
  - Example of a function Y(t) with
    - Taylor polynomial 1+t: once differentiable at t=0
    - Hoelder polynomial 1+t+t^2: best polynomial approximation
    - Regularity 3.5
  - $Y(t) \;=\; 1 + t + t^2 + t^{3.5} \sin(1/t)$

$$\begin{aligned} Y'(t) &= 1 + 2t + 3.5t^{2.5} \sin(1/t) - t^{1.5} \cos(1/t) \\ Y''(t) &= 2 + 3.5 \cdot 2.5t^{1.5} \sin(1/t) + ... + t^{-.5} \sin(1/t) \end{aligned}$$

Direct link via fractional wavelets

- Consider fractional Wavelets defined in frequency:

$$\Psi_\nu(u) = c|u|^\nu \exp(-u^2) \geq 0$$

- Lemma: If either side of the following exists then

$$\mathrm{Sup}_a\, T_\nu(a,0)\, a^{-\nu} = c\, E[\, |X|^\nu\,]$$

Proof:

$$\begin{aligned}\underline{T_\nu(a,0)a^{-\nu}} &= a^{-\nu}\,\frac{1}{a}\int \phi_X(u)\psi_\nu(u/a)du \\ &= a^{-\nu}\int \Psi_\nu(ax)dF_X(x) \\ &= c\int |x|^\nu \exp(-(ax)^2)dF_X(x) \overset{a\to 0}{\to} \underline{c\int |x|^\nu dF_X(x)}\end{aligned}$$

Parseval

Monotone convergence

- Fill `gap’ of Lukacs/Ramachandran

$$\mathcal{R}e\ \phi(u) - \sum_{k=1}^{p} a_{2k}u^{2k} = O(|u|^\lambda) \Rightarrow \mathbb{E}|X|^{2p} < \infty$$

Summary: Char. Function 201

- Let 2p<λ≤2p+2 with integer p.
  - Let a_k be any coefficients

$$\boxed{\mathbb{E}[|X|^r] < \infty \quad \text{for all } r < \lambda}$$

$$\Leftrightarrow \boxed{\mathcal{R}e\ \phi(u) - \sum_{k=1}^{p} a_{2k} u^{2k} \overset{u \to 0}{=} O(|u|^r) \quad \text{for all } r < \lambda}$$

$$\Leftrightarrow \boxed{T(a,0) = \mathbb{E}[|\Psi(aX)|] \overset{a \to 0}{=} O(|a|^r) \quad \text{for all } r < \lambda}$$

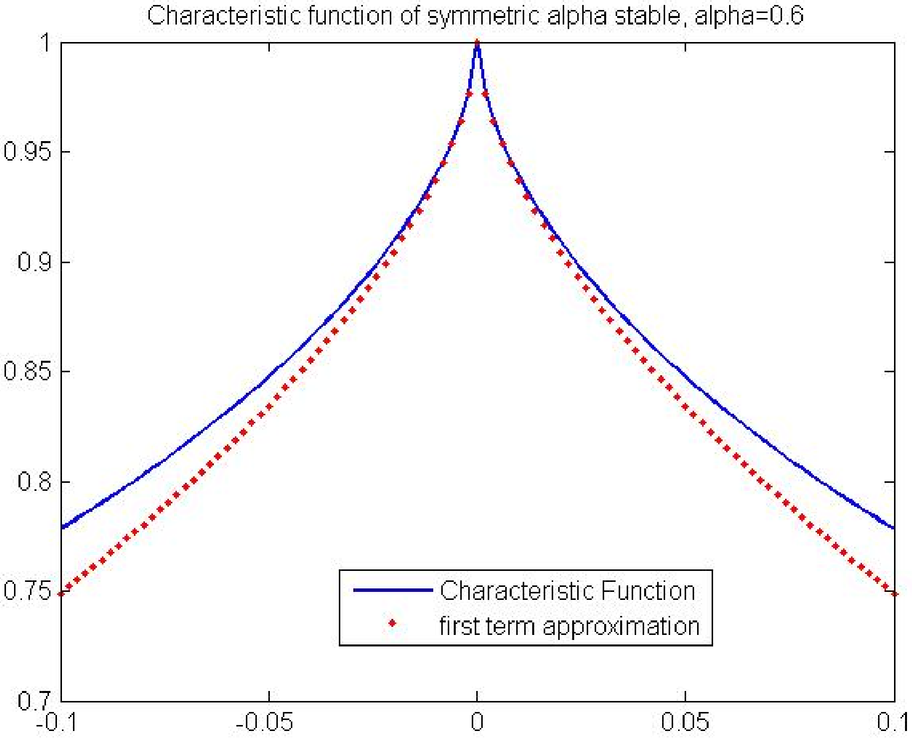
Implementation
and
Performance

Numerical demonstration

Characteristic function of stable law at the origin

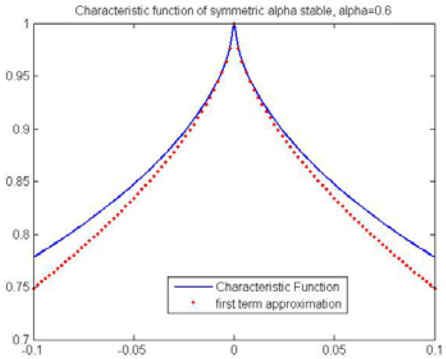
$$\phi(u) = \mathcal{R}e\ \phi(u) \quad = \quad \underline{\exp(-|u|^{\alpha})} \simeq \underline{1 - |u|^{\alpha}} + \ldots \qquad (|u| \to 0)$$

$\alpha = 0.6$



Numerical demonstration

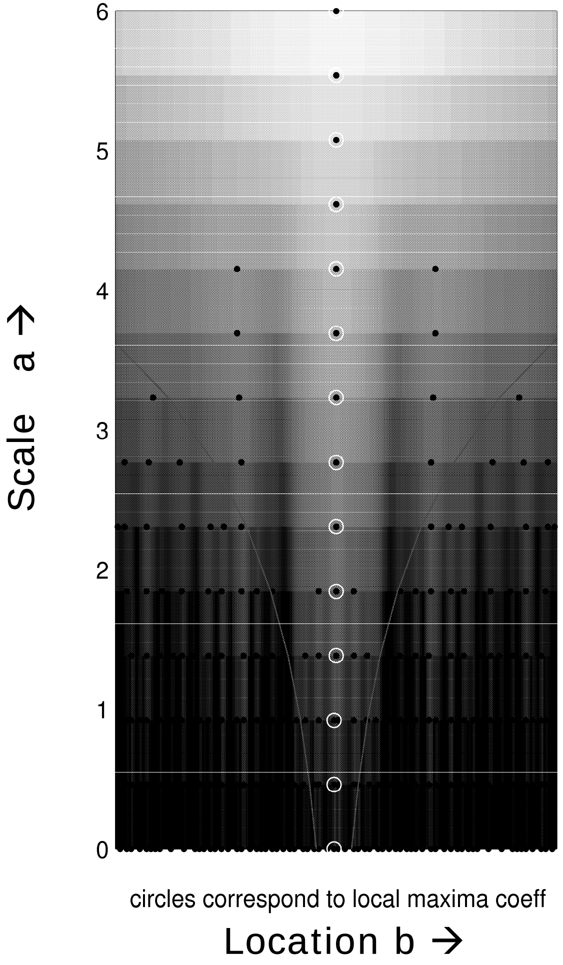
Characteristic Function



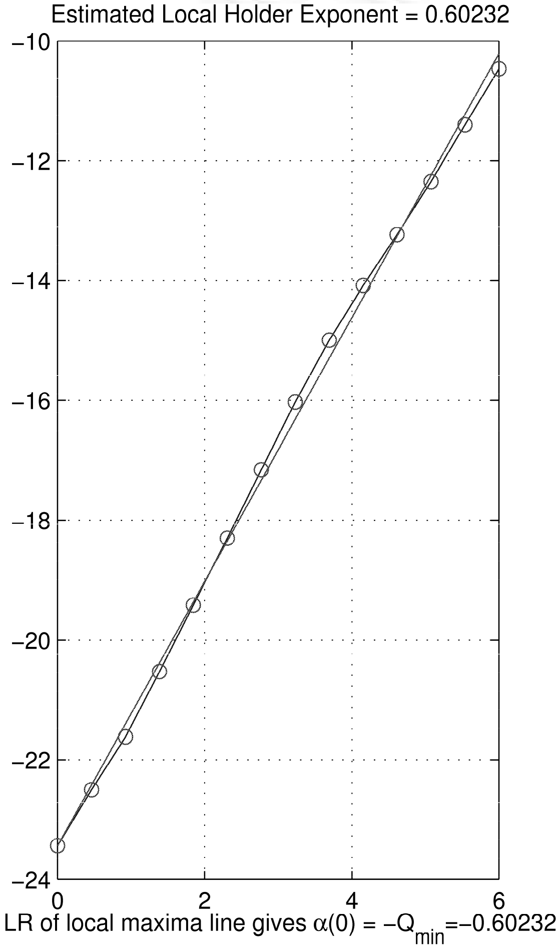
$\alpha = 0.6$

Wavelet Transform

$$T(a,b) = \int \mathcal{R}e\ \phi(t) \cdot \frac{1}{a}\psi\left(\frac{t-b}{a}\right) dt$$



Estimation of scaling exponent



Numerical Implementation

$$|T(a,0)| = \mathbb{E}[|\Psi(aX)|] \sim a^{\lambda}$$

The estimator of T(a,0) of ϕ is

- …simple:

$$\widehat{T}(a,0) = \widehat{\mathbb{E}}[\Psi(aX)] = 1/N \sum_{k=1}^{N} \Psi(aX_k)$$

- …unbiased
- …non-parametric!

- Estimation of critical order λ = sup{q: E[|X|^q]<∞ }

$$\boxed{1/N \sum_{k=1}^{N} \Psi(aX_k) \simeq a^{\lambda} \quad \text{as } a \to 0}$$

Practical Considerations

$$1/N \sum_{k=1}^{N} \Psi(aX_k) \simeq a^{\lambda} \quad \text{as } a \to 0$$

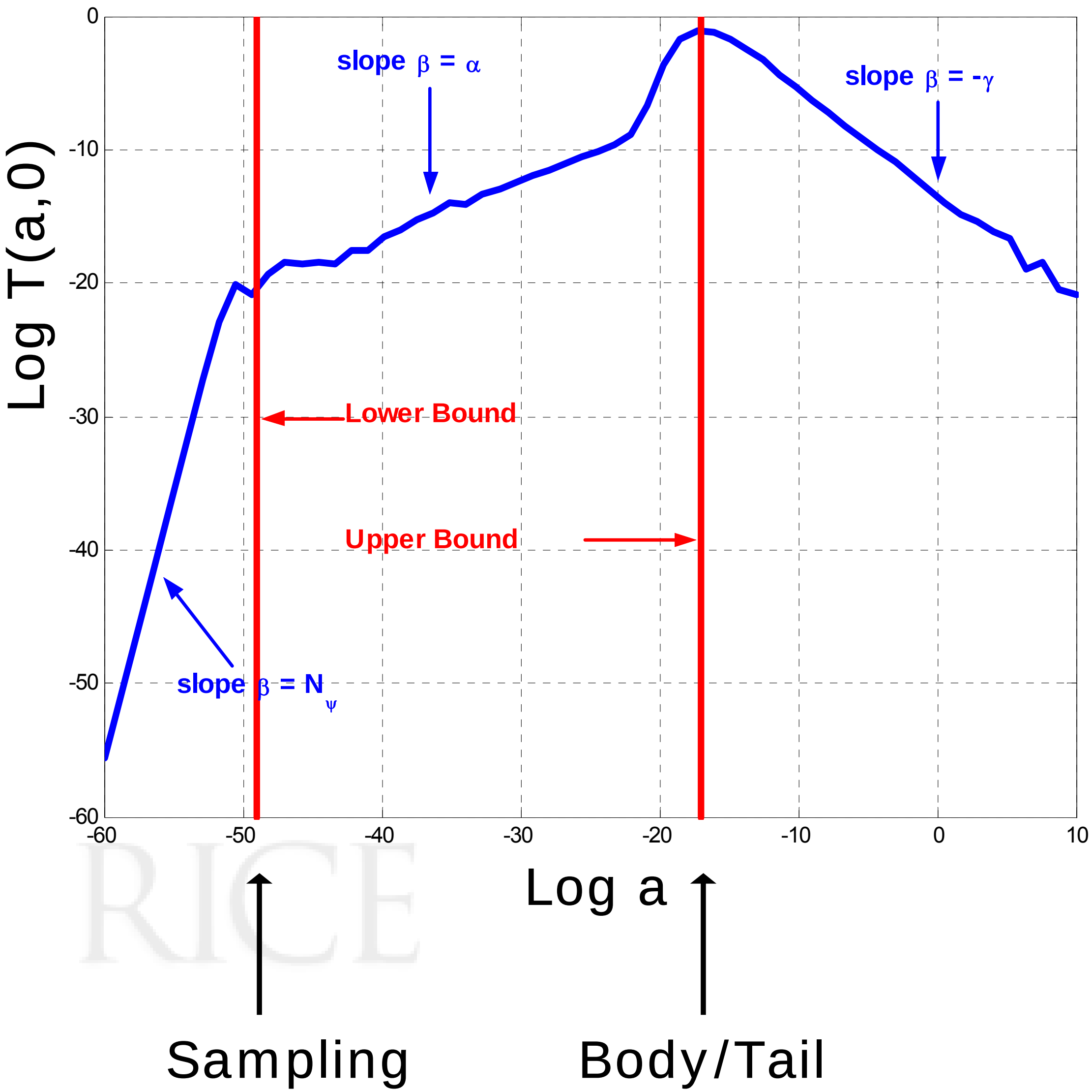
- Choose a wavelet
  - With high enough regularity (N>λ)
  - With real positive Fourier transform (ex: even derivatives of Gaussian kernel)
- Cutoff scales J0 < j < J1
  - Shannon argument on max $\{x_i\}$: lower bound J0
  - Body / Tail frontier : upper bound J1
- Interpretation of estimator:
  - Weight-average of samples with weight Ψ(aX)
  - Shift weights out to large samples by scaling a→0

Cutoff scales

Ex: Hybrid distribution
(Gamma body and stable tails)

- (for $x >= \delta$)
  - $x \sim \alpha$-stable (β=1),
  - E $|x|^r = \infty$, r >= α

- (for $x < \delta$)
  - $x \sim \Gamma(\gamma)$
  - E $|x|^r = \infty$, r <= $-\gamma$

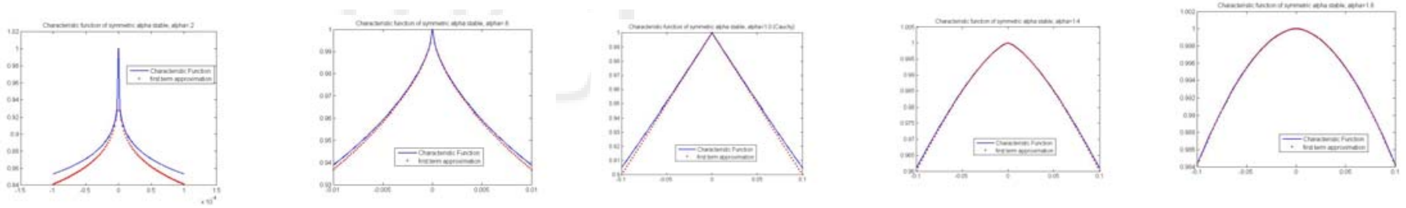


Competing for stable parameter

Alpha-stable Laws:

- compare with Koutrouvelis’80 and McCullogh’86 are parametric (stable distribution)
- non-parametric wavelet based estimator is
  - competitive
  - especially for intermediate to small a

α	0.2	0.6	1	1.4	1.8
Wavelet based	0.196 ± 0.007	0.58 ± 0.018	1.0 ± 0.035	1.46 ± 0.066	1.74 ± 0.02
$\widehat{\alpha}$ (Koutrouvelis)	ND	0.60 ± 0.007	1.0 ± 0.009	1.403 ± 0.013	1.80 ± 0.012
$\widehat{\alpha}$ (McCullogh)	0.59 ± 0.0018	0.605 ± 0.009	1.0 ± 0.009	1.40 ± 0.016	1.80 ± 0.022



Competing for Pareto parameter

1/Gamma Laws:

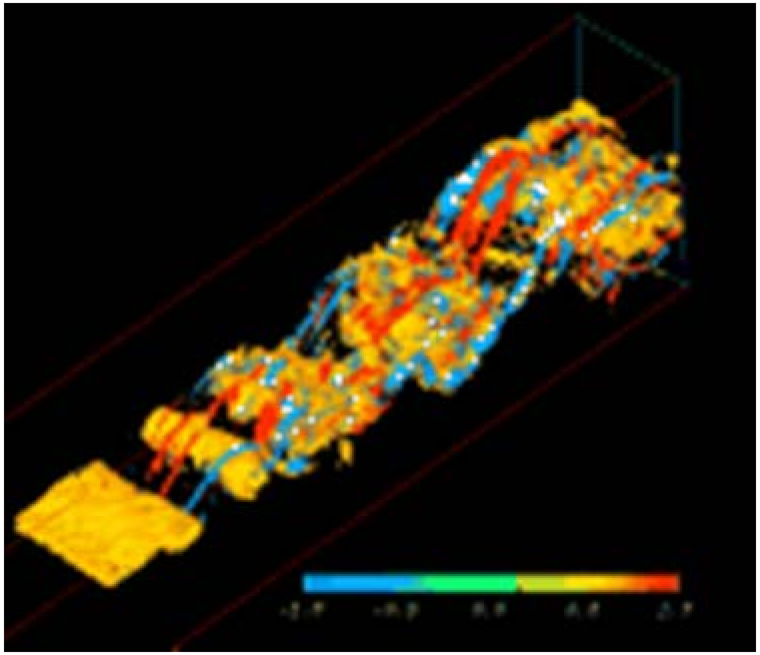
- Pareto
- Koutrouvelis'80 and McCullogh'86 are parametric (stable distribution)
- non-parametric wavelet based estimator is
  - superior

γ	0.2	0.4	0.6	0.8
Wavelet based	0.204 ± 0.007	0.395 ± 0.008	0.589 ± 0.015	0.793 ± 0.03
$\widehat{\alpha}$ (Koutrouvelis)	ND	0.433 ± 0.006	0.56 ± 0.007	0.67 ± 0.009
$\widehat{\alpha}$ (McCullogh)	0.513 ± 0.000	0.514 ± 0.000	0.583 ± 0.009	0.72 ± 0.013

Model identification

…through scaling of moments

Why Moments and Scaling



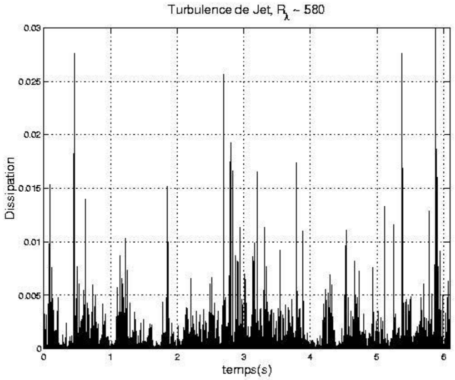
Turbulence: models wanted

- Velocity field v(x)

- Kolmogorov 1941:
  - $\mathbb{E}|v(t+\delta)-v(t)|^q \simeq \delta^{q/3}$
  - Linear model, fBm

- Kolmogorov 1962:
  - $\mathbb{E}|v(t+\delta)-v(t)|^q \simeq \delta^{\tau(q)}$
  - Multiplicative model, Cascade

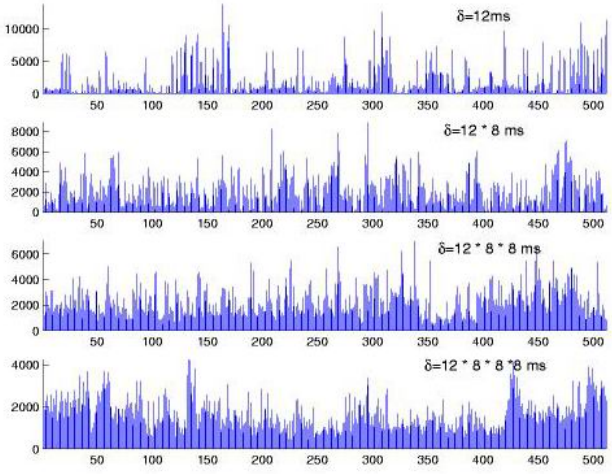
Courtesy P. Chainais

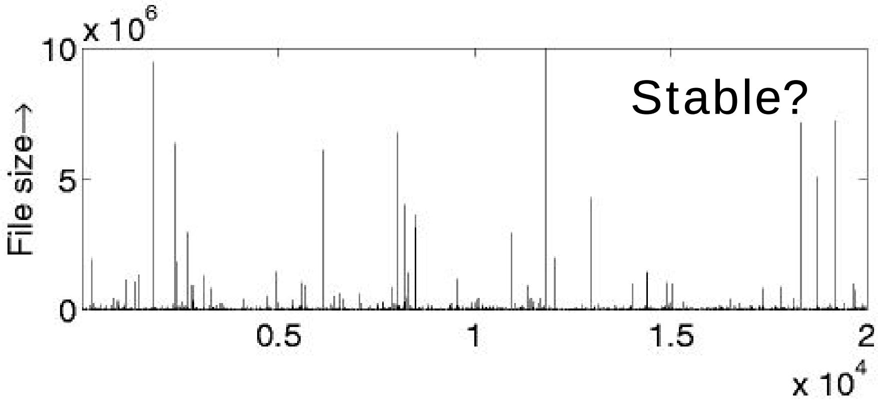


Scaling and statistical aspects

- Networks
  - Non-Gaussianity / Long-memory
  - Model identification (cascade?)
- WWW
  - File size distribution
- Stock Markets
  - Long-memory

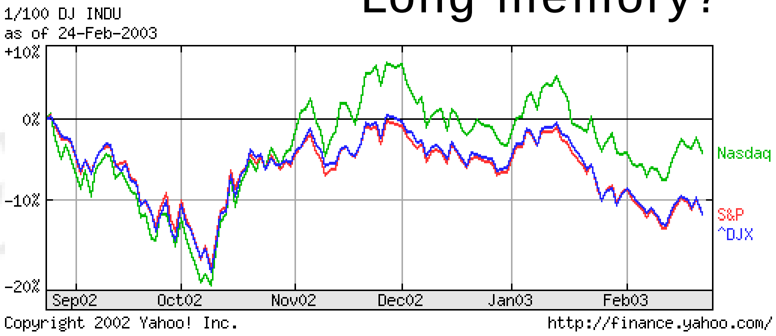
Log-normal?





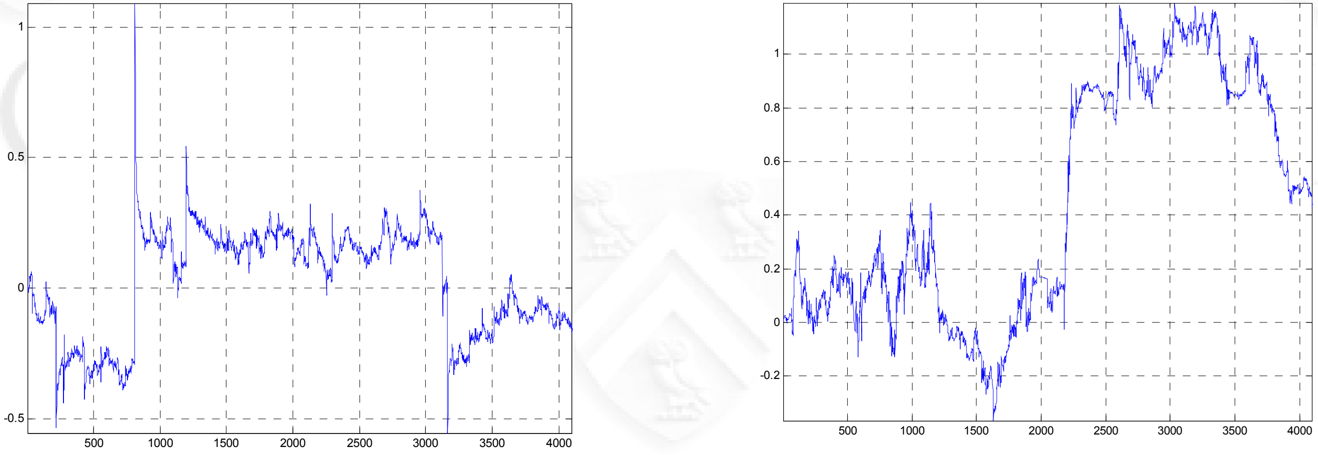
Stable?

Long memory?



Identify the Multifractal

- One of these signals is a stable Levy flight,
- …the other is a multiplicative cascade.
- Which is which?

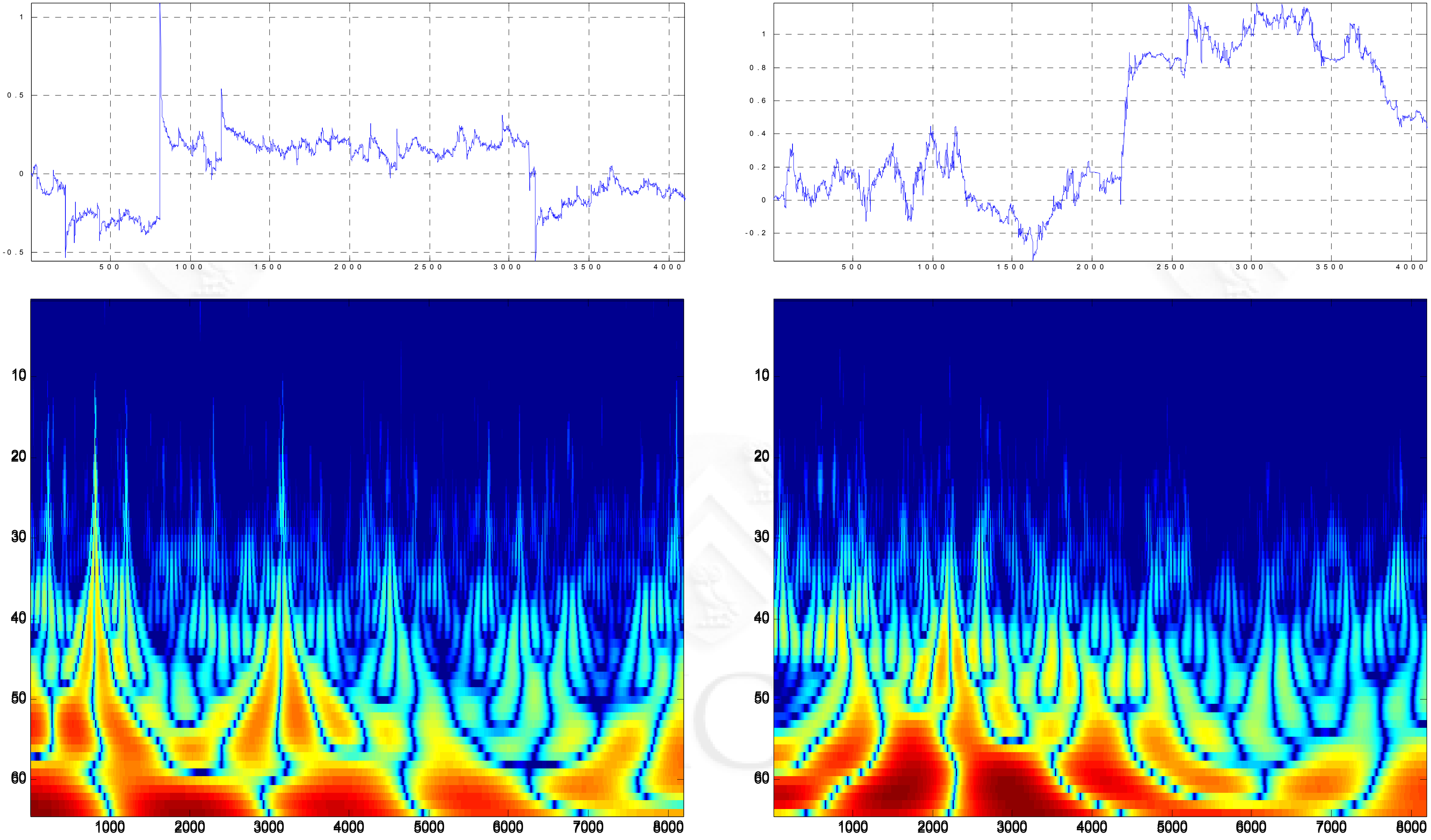


- Note:

Self-similar Levy flight: $\mathbb{E}[|L(t+\delta)-L(t)|^q] \simeq \delta^{qH}$

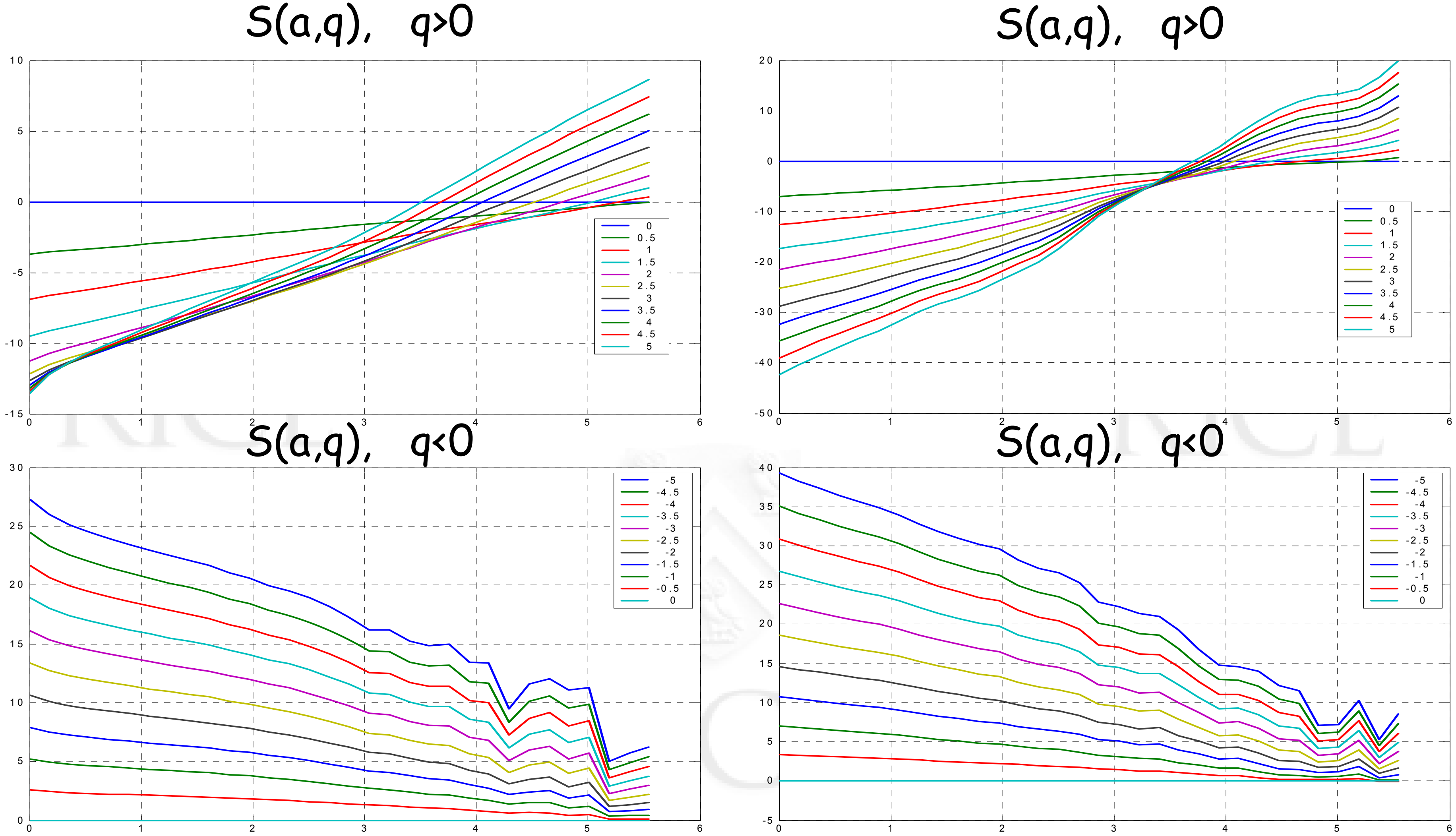
Multiplicative Cascade: $\mathbb{E}[|M(t+\delta)-M(t)|^q] \simeq \delta^{\tau(q)}$

Wavelet transform: $a^{-\frac{1}{2}} \int x(t)\, \psi((t-b)/a)\, dt$



Challenge: which wavelet to use.

Estimating $S(a,q) := \mathbb{E}|T(a,b)|^q$



S(a,q), q>0

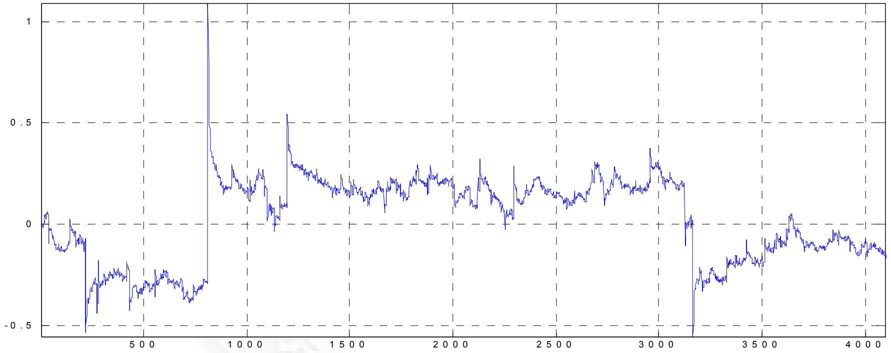
S(a,q), q>0

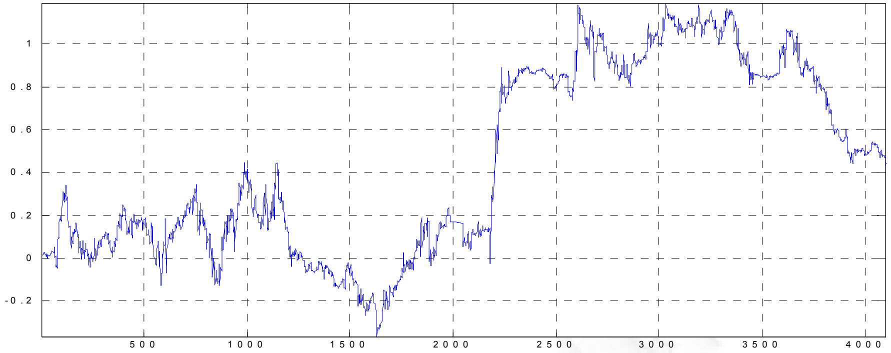
S(a,q), q<0

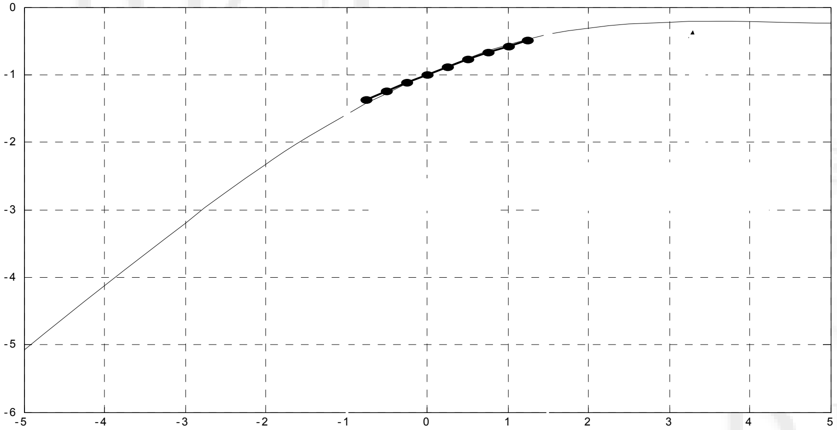
S(a,q), q<0

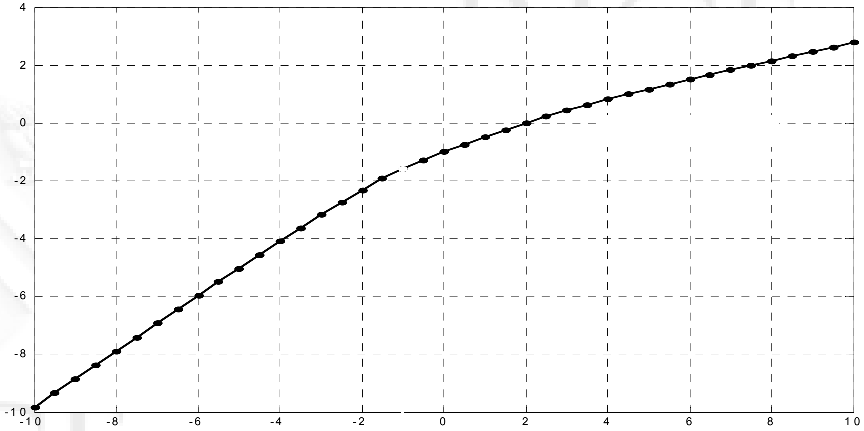
Challenge: which orders to use.

Estimate of τ from $S(a,q) := \mathbb{E}|T(a,b)|^q \simeq a^{\tau(q)}$



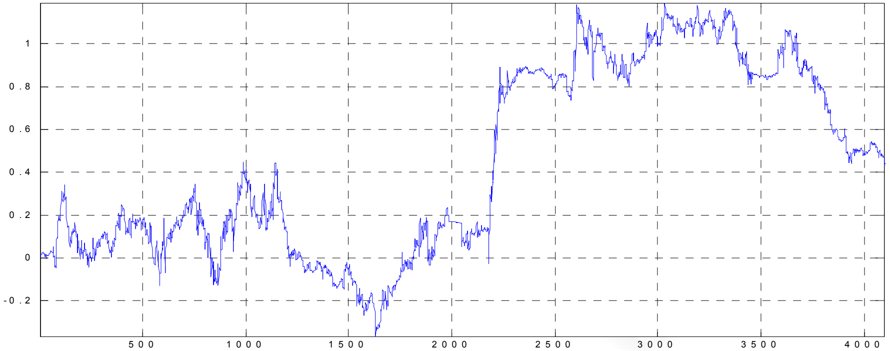


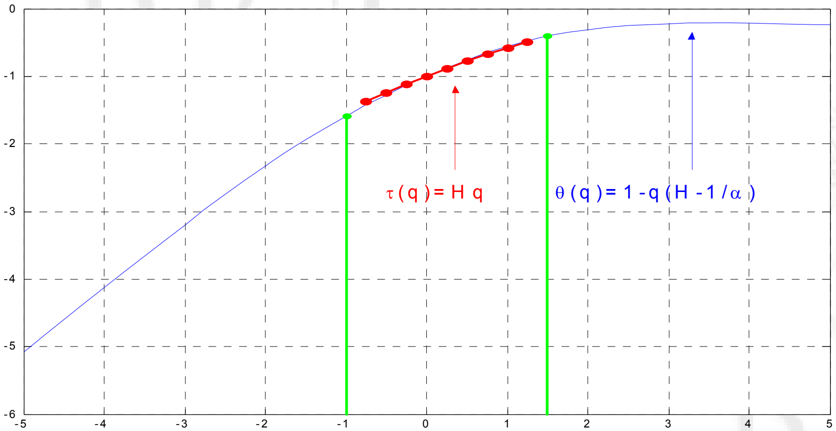


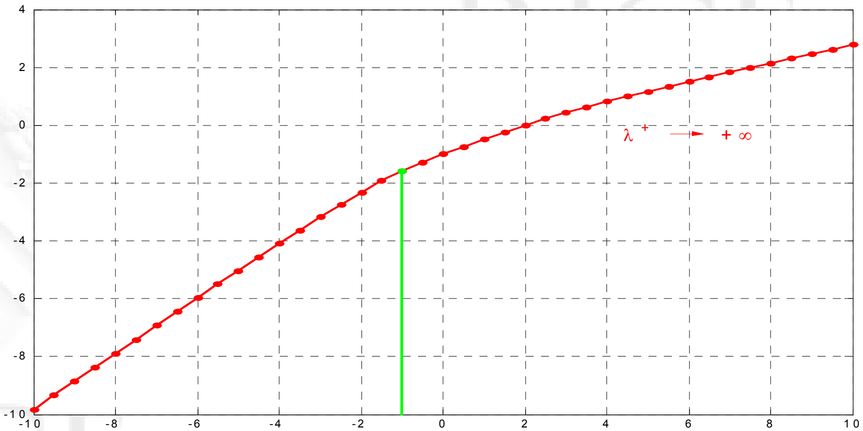


Challenge: Interpretation.

Supervised moment estimation







The moments exist only for a few q. The linear τ(q) hints to a selfsimilar process (Levy flight)

The moments exist in a wide range. The non-linear τ(q) hints to a multiplicative cascade.

Summary

- Wavelets useful for non-parametric estimation
- Holder regularity of characteristic function tied to existence of moments beyond order 2
- Estimating critical order of finite moments useful for
  - – Tail estimation
  - – Model identification

References: Scaling processes

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- Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance*, Chapman and Hall.

- Doukhan, Oppenheim and Taqqu (eds) (2002): *Long range dependence : theory and applications,* Birkhaeuser

- Software:
  - Goncalves: http://www.inrialpes.fr/is2/people/pgoncalv
  - Veitch: http://www.emulab.ee.mu.oz.au/~darryl
  - Riedi: http://www.stat.rice.edu/~riedi

The end

Papers (JASA, TechRep)

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