Modeling Infinitely Divisible Scaling: beyond Powerlaws

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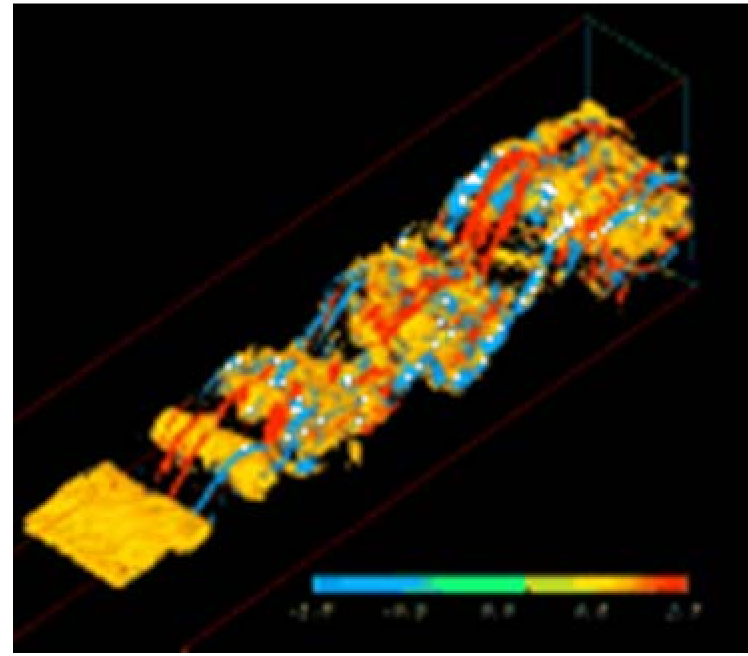
Scaling Analysis

An empirical view

Why Cascades



Turbulence: models wanted

- Kolmogorov 1941 :

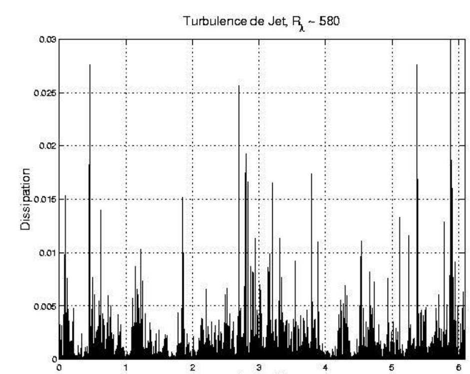
$< [v(x+r)-v(x)]^q > \sim r^{q/3}$

- Kolmogorov 1962 :

$< [v(x+r)-v(x)]^q > \sim r^{H(q)}$

- Data is non-Gaussian
- ...presents structure on all scales

Courtesy P. Chainais

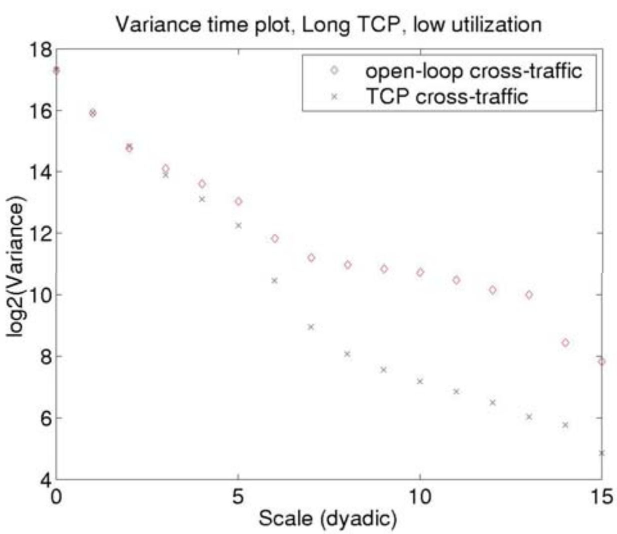


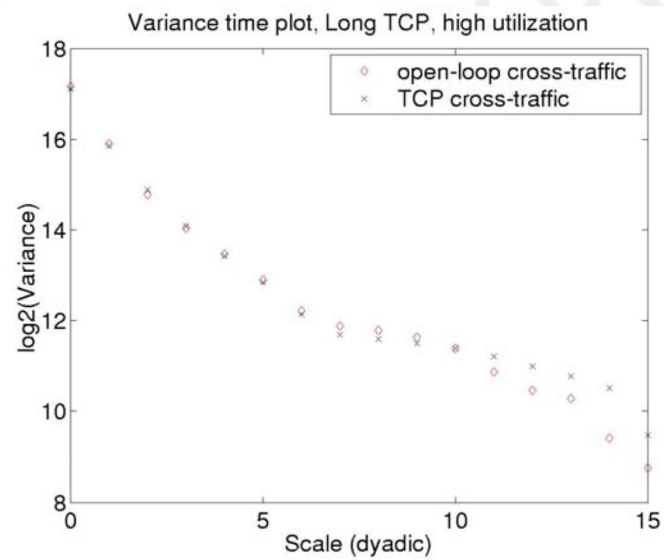
Powerlaws?

Real world data

- can deviate from powerlaws: network traffic
- Lukacs:

if the data does not fit to the model then too bad for the data.





Infinitely divisible scaling

$$\text{Self-similarity:} \quad \mathbb{E}[|B(t+\delta) - B(t)|^q] \simeq \delta^{qH}$$

$$\text{Multifractal scaling:} \quad \mathbb{E}[|M(t+\delta) - M(t)|^q] \simeq \delta^{\tau(q)}$$

$$\text{IDC scaling:} \quad \mathbb{E}[|X(t+\delta) - X(t)|^q] \simeq \exp[n(\delta)\zeta(q)]$$

- Multifractal scaling reduces to self-similarity if τ is linear in q. (sometimes called mono-fractal)
- IDC reduces to multifractal scaling if $n(\delta)=-\log(\delta)$
  - thus, for powerlaws
- In general $n(\delta)$ gives the speed of the cascade

- IDS suggested as a framework for statistical analysis in turbulence [Castaing] ...but where are the models?
- Greatest potential for models based on multiplication

Multiplicative models

- In distribution, the increment X(t+δ)-X(t) looks like a n(δ)-fold product of iid multipliers

$$
\begin{aligned}
\text{IDS: } \mathbb{E}[e^{q \log |X(t+\delta)-X(t)|}] &= \mathbb{E}[|X(t+\delta)-X(t)|^q] \\
&\simeq \exp[n(\delta)\zeta(q)] \\
&= (e^{\zeta(q)})^{n(\delta)}
\end{aligned}
$$

- Since n(δ) is a function of scale, multipliers are over scale not over time

- Warning: this does not mean that there are actual multipliers such that X(t+δ)-X(t) = $M_1 \ldots M_{n(\delta)}$ a.s. :
  - For Brownian motion these n factors would have to be constant which is non-sense (B(t) would be a.s. a square root)

$$B(2^{-n}) \stackrel{d}{=} (1/\sqrt{2})^n B(1)$$

Infinitely Divisible Cascades

Intuition:

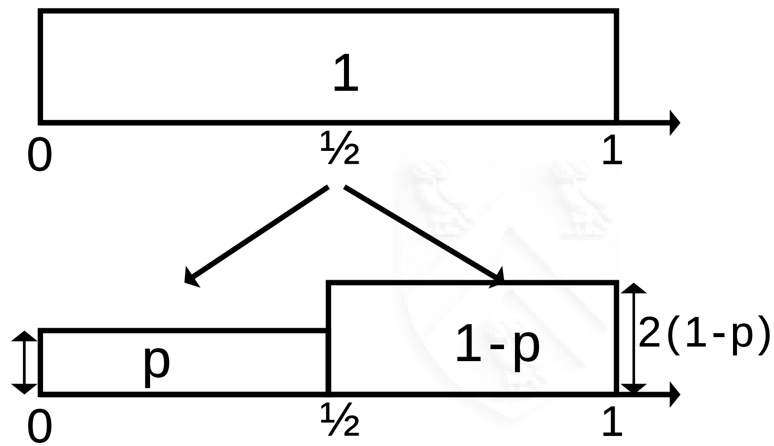
A modeling framework for ID-scaling

Toy: Binomial cascade

$$\epsilon := \{\epsilon_k\}_k \in \{0,1\}^{\mathbb{N}}$$

$$l_n(\epsilon) := \#\{k \le n : \epsilon_k = 1\} = \sum_{k=1}^{n} \epsilon_k$$

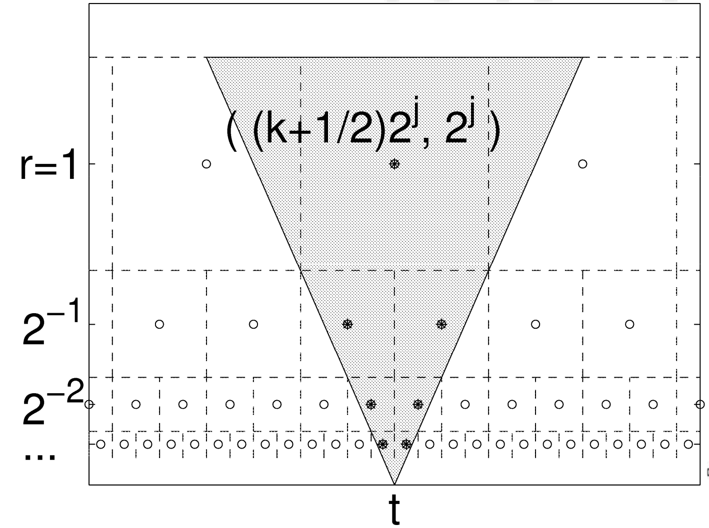
$$I_n(\epsilon) := [\sum_{k=1}^{n} \epsilon_k/2^k, \sum_{k=1}^{n} \epsilon_k/2^k + 1/2^n))$$



Mass re-distribution

$$\mu_n(I_n(\epsilon)) \;=\; p^{n-l_n(\epsilon)}(1-p)^{l_n(\epsilon)}$$

$$\mu_m(I_n(\epsilon)) \;=\; \mu_n(I_n(\epsilon)) \quad m \ge n$$
$$\rightarrow \;\; \mu(I_n(\epsilon))$$



Binomial as a product of pulses

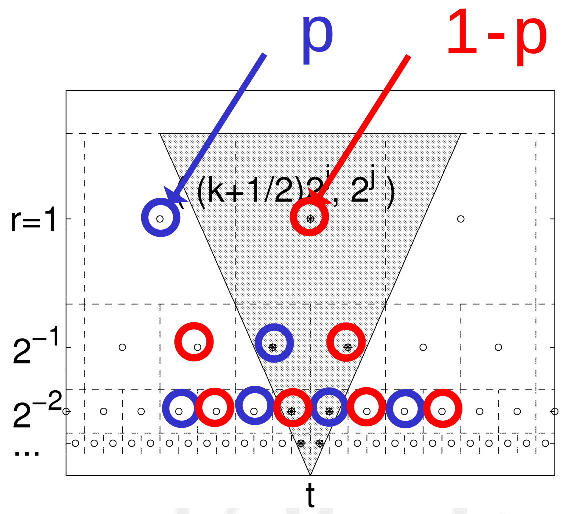
We may write

$$\mu(I_n(\epsilon)) \;=\; \int_{I_n(\epsilon)} Q_n(t)dt$$

where

$$Q_n(t) \;=\; 2^n p^{n-l_n}(1-p)^{l_n}$$

and $l_n(\epsilon(t))$ is constant over each I_n.



Assign multipliers $2p$ and $2(1-p)$ to dyadic intervals and compute $Q_n(t)$ as the product of multipliers contained in the cone

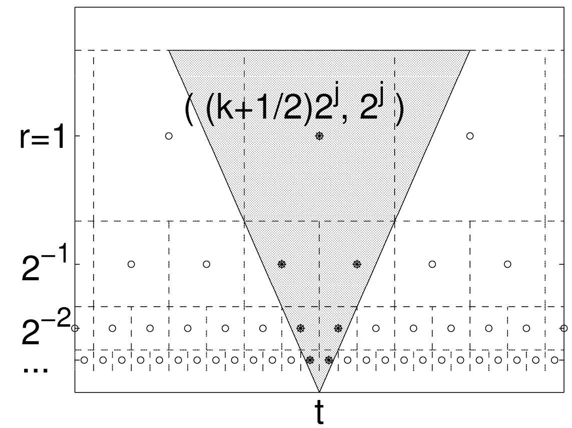
$$C_r(t) = \{(t_i, r_i) : t - r_i/2 < t_i < t + r_i/2, r_i > r\}$$

From Binomial to IDC

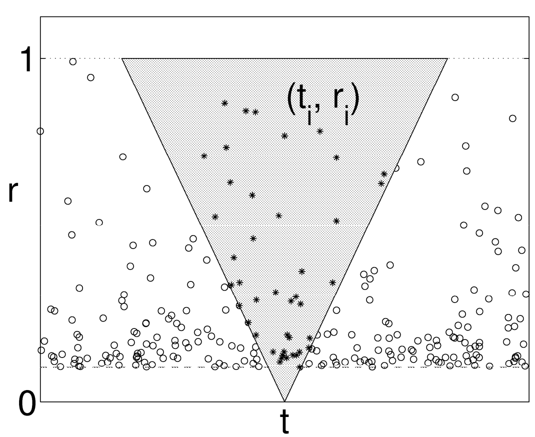
We may write

$$\mu(I_n(\epsilon)) \;=\; \int_{I_n(\epsilon)} Q_n(t)\,dt$$

where $Q_n(t)$ is a product of multipliers.

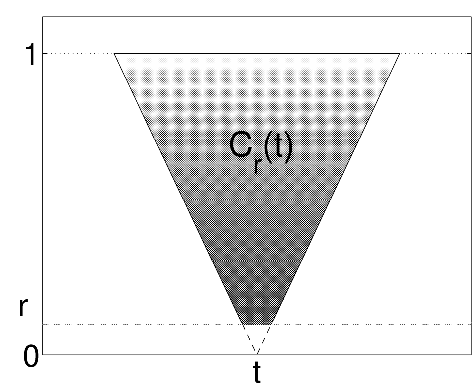


- Assign multipliers according to
  - a marked point process...
  - ...or a random measure



$$\prod_{(t_i,r_i)\in C_r(,t)} W_i$$

$$\exp[M(C_r(t))]$$



Infinitely Divisible Cascades

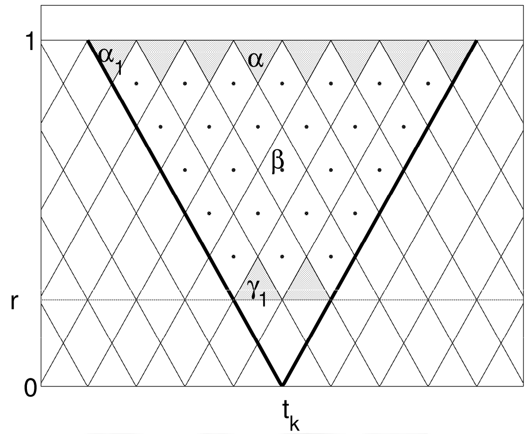
Definitions:

Infinitely divisible measures

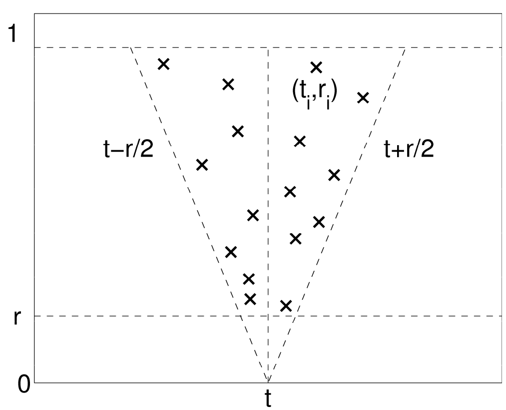
Cascades

Infinitely Divisible Measure

- M is an infinitely divisible measure
  - Randomly scattered:
    - M(E), M(F) independent for disjoint E and F
  - Measure of set E is inf. div. r.v.



$$\mathbb{E}[e^{qM(E)}] = \exp\left[-\rho(q)m(E)\right]$$



Ex 1: Poisson count measure:

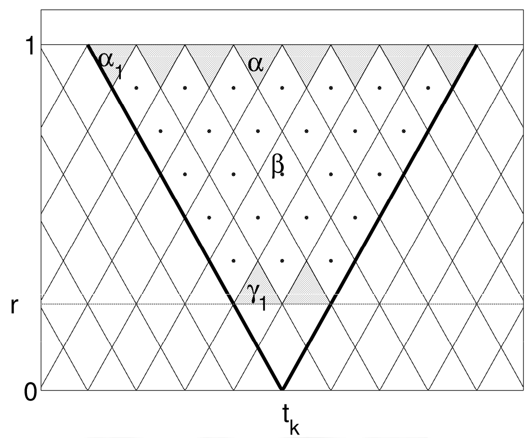
M(E) = Poisson variable with mean m(E)

$$\mathbb{E}[e^{qM(E)}] = \sum_{k\geq 0} e^{-m(E)}\frac{m(E)^k}{k!}e^{qk} = \exp\left[(e^q - 1)m(E)\right]$$

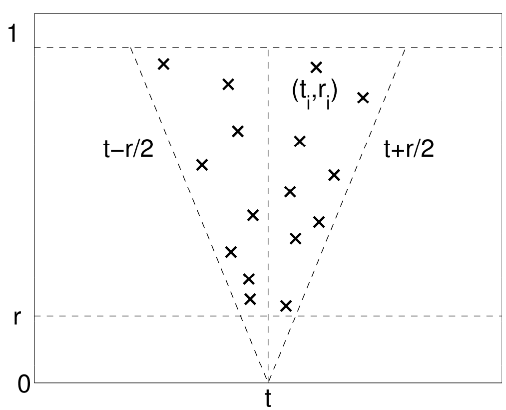
$$\rho(q) = 1 - e^q$$

Infinitely Divisible Measure

- M is an infinitely divisible measure
  - Randomly scattered:
    - M(E), M(F) independent for disjoint E and F
  - Measure of set E is inf. div. r.v.



$$\mathbb{E}[e^{qM(E)}] = \exp\left[-\rho(q)m(E)\right]$$



Ex 1: Poisson count measure:

M(E) = Poisson variable with mean m(E)

$$\rho(q) = 1 - e^q$$

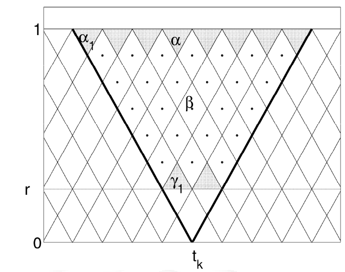
Ex 2: Poisson process, marked with W

M(E) = Compound Poisson, marks W, mean m(E)

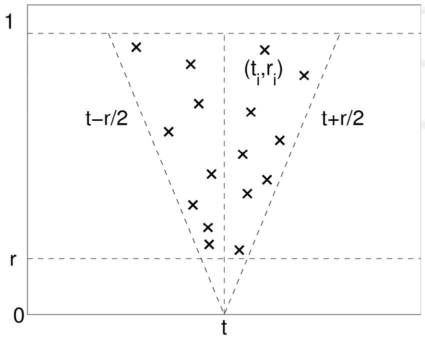
$$\rho(q) = 1 - \mathbb{E}[W^q]$$

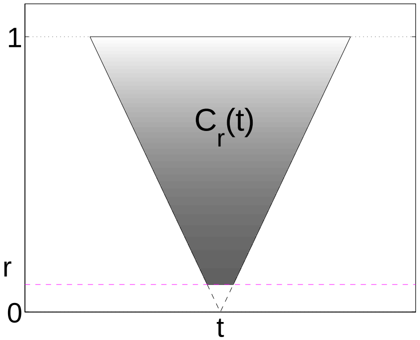
Infinitely Divisible Measure

- M is an infinitely divisible measure
  - Randomly scattered:
  - Measure of set E is inf. div. r.v.



$$\mathbb{E}[e^{qM(E)}] = \exp\left[-\rho(q)m(E)\right]$$





Ex 1: Poisson: mean m(E)

$$\rho(q) = 1 - e^q$$

Ex 2: Compound Poisson: mean m(E), mark W

$$\rho(q) = 1 - \mathbb{E}[W^q]$$

Ex 3: Gaussian measure: variance factor m(E)

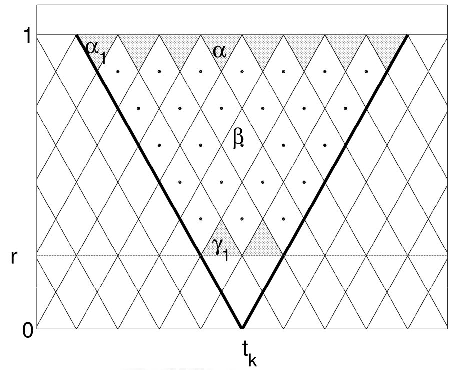
(μ=0 w.l.o.g. due to later normalization)

$$\rho(q) = -q\mu - q^2\sigma^2/2$$

Infinitely Divisible Cascade

- M is an infinitely divisible measure
  - Randomly scattered
  - Measure of set E is inf. div. r.v.

$$\mathbb{E}[e^{qM(E)}] = \exp\left[-\rho(q)m(E)\right]$$



  - Assume exp(M(E)) has finite mean (ρ(1) defined)
- Parameters:
  - Infinitely divisible law via ρ(q)
  - control measure dm(t,r)
  - Cone: causal $C_r(t) = \{(t_i, r_i) : t - r_i < t_i < t, r_i > r\}$

    symmetrical $C_r(t) = \{(t_i, r_i) : t - r_i/2 < t_i < t + r_i/2, r_i > r\}$

- IDC: $\tilde{Q}_r(t) \;=\; \exp[M(C_r(t))] \qquad Q_r(t) \;=\; \dfrac{\tilde{Q}_r(t)}{\mathbb{E}[\tilde{Q}_r(t)]}$

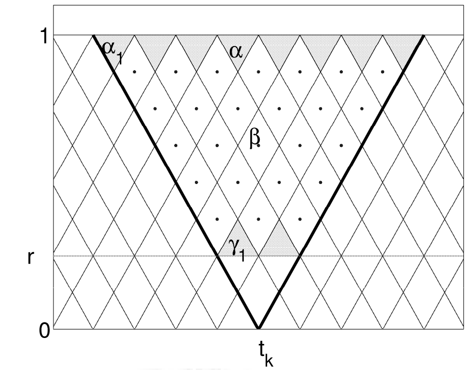
Infinitely Divisible Cascades

1st Properties:
Scaling

Scaling of Infinitely Divisible Cascades

- M is an infinitely divisible measure
  - Randomly scattered
  - Measure of set E is inf. div. r.v.

$$\mathbb{E}[e^{qM(E)}] = \exp\left[-\rho(q) m(E)\right]$$



- IDC: $\tilde{Q}_r(t) = \exp[M(C_r(t))]$ $\qquad Q_r(t) = \dfrac{\tilde{Q}_r(t)}{\mathbb{E}[\tilde{Q}_r(t)]}$

$$\frac{\mathbb{E}[e^{qM(E)}]}{\mathbb{E}[M(E)]^q} = \exp\left[-\underbrace{(\rho(q) - q\rho(1))}_{\varphi(q)} m(E)\right]$$

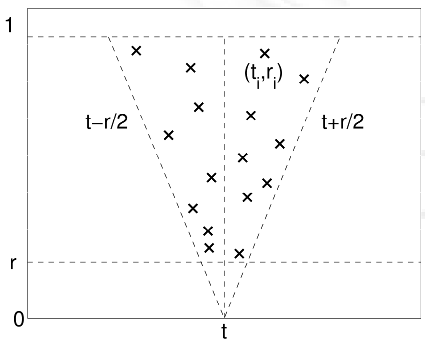
- Scaling: $\mathbb{E}[Q_r(t)^q] = \exp[-\varphi(q) m(C_r(t))]$

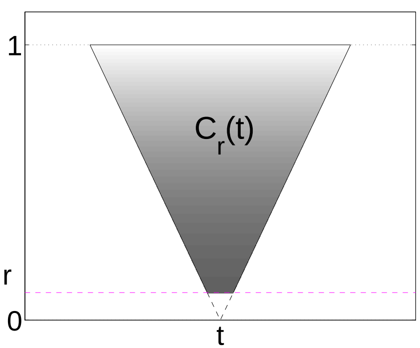
Self-similarity:	$\mathbb{E}[\lvert B(t+\delta) - B(t)\rvert^q]$	$\simeq$	δ^{qH}
Multifractal scaling:	$\mathbb{E}[\lvert M(t+\delta) - M(t)\rvert^q]$	$\simeq$	$\delta^{\tau(q)}$
IDC scaling:	$\mathbb{E}[\lvert X(t+\delta) - X(t)\rvert^q]$	$\simeq$	$\exp[n(\delta)\zeta(q)]$

$m(C_r) \rightarrow n(\delta)$, speed
$\varphi(q) \rightarrow \zeta(q)$, law

Infinitely Divisible Cascade: examples

- **IDC Scaling:** $$\mathbb{E}[Q_r(t)^q] = \exp\left[-\underbrace{(\rho(q) - q\rho(1))}_{\varphi(q)} m(C_r(t))\right]$$
  - Note: $\varphi(0) = \varphi(1) = 0$ and φ is convex





Ex 1: Poisson: mean m(E)

$$\rho(q) = 1 - e^q \qquad \varphi(q) = 1 - e^q + q(e-1)$$

Ex 2: Compound Poisson: mean m(E), mark W

$$\rho(q) = 1 - \mathbb{E}[W^q] \qquad \varphi(q) = 1 - \mathbb{E}[W^q] - q(1 - \mathbb{E}[W])$$

$$\varphi(2) = -\mathbb{E}[(W-1)^2]$$

Ex 3: Gaussian measure: variance factor m(E)

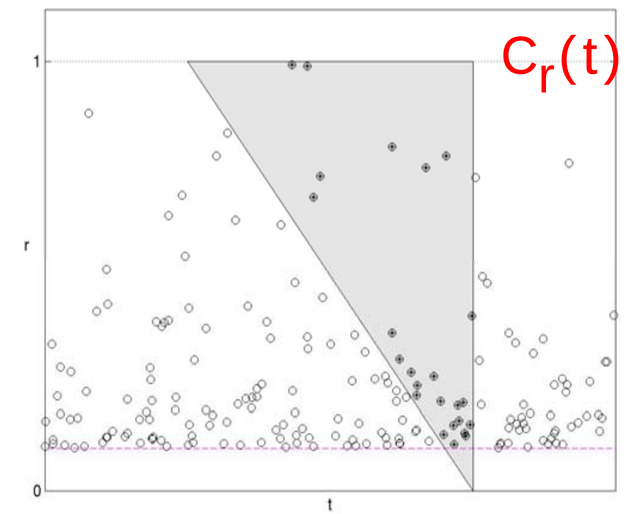
$$\rho(q) = -q\mu - q^2\sigma^2/2 \qquad \varphi(q) = -(q^2 - q)\sigma^2/2$$

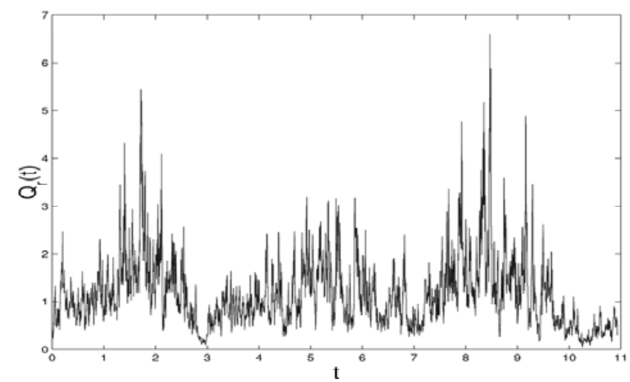
Ex 4: Stable, $\alpha<1$, $\beta=-1$

$$\varphi(q) = \sigma^\alpha(q - q^\alpha)\left(1 - \tan\left(\frac{\pi\alpha}{2}\right)\right) \qquad \text{for } q > 0.$$

Special case: CPC

- Marked Poisson process (T_i, R_i, W_i)
  - m(A) = E[#points in A]
  - Marks W_i are i.i.d.





Cascade Process at scale r:

$$Q_r(t) = \gamma_r \cdot \prod_{(t_i,r_i)\in C_r(,t)} W_i$$

$$\gamma_r = 1/\mathbb{E}\left[\prod_{(t_i,r_i)\in C_r(t)} W_i\right]$$

CPC: dual view

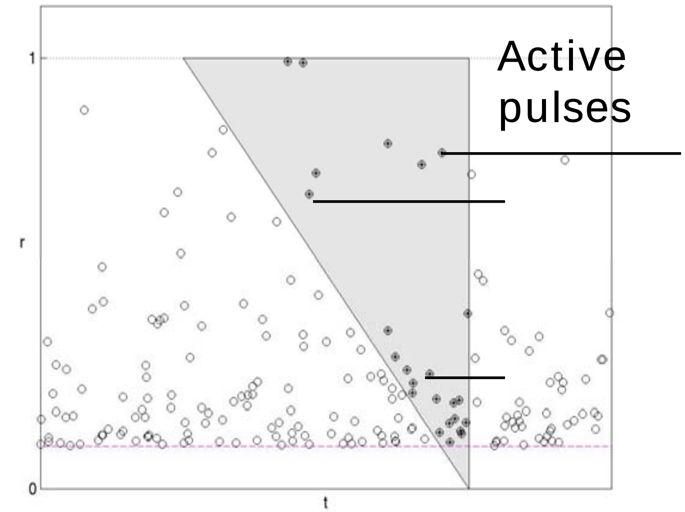
- Pulse at (T_i, R_i, W_i)

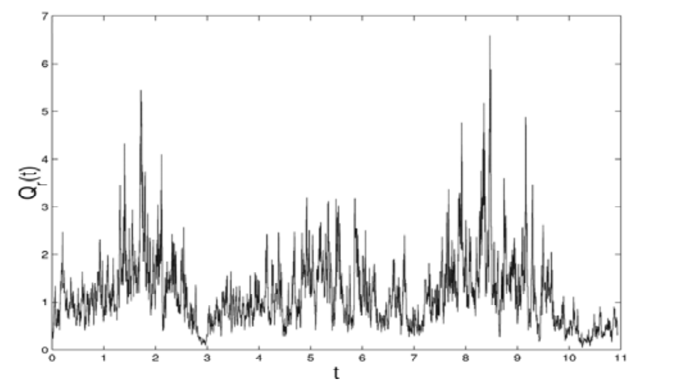
$$P_i(t) = \begin{cases} W_i & \text{if } T_i \le t < T_i + R_i \\ 1 & \text{else} \end{cases}$$

- The pulse multipliers active at time t are the ones with (T_i, R_i) in the cone
- Cascade Process at scale r:

$$Q_r(t) = \gamma_r \cdot \prod_{r_i > r} P_i(t)$$

- Compare with infinite Poisson source model from queuing





Infinitely Divisible Cascades

First look at parameters:

Control measure

- Stationary increments
- Scaling

Stationarity of Q(t)

- IDC defined as

$$Q_r(t) \;=\; \frac{\tilde{Q}_r(t)}{\mathbb{E}[\tilde{Q}_r(t)]} \qquad \tilde{Q}_r(t) \;=\; \exp[M(C_r(t))]$$

- Recall

$$\mathbb{E}[e^{qM(E)}] = \exp\left[-\rho(q)m(E)\right]$$

- Stationary if m is time-translation invariant, e.g.:

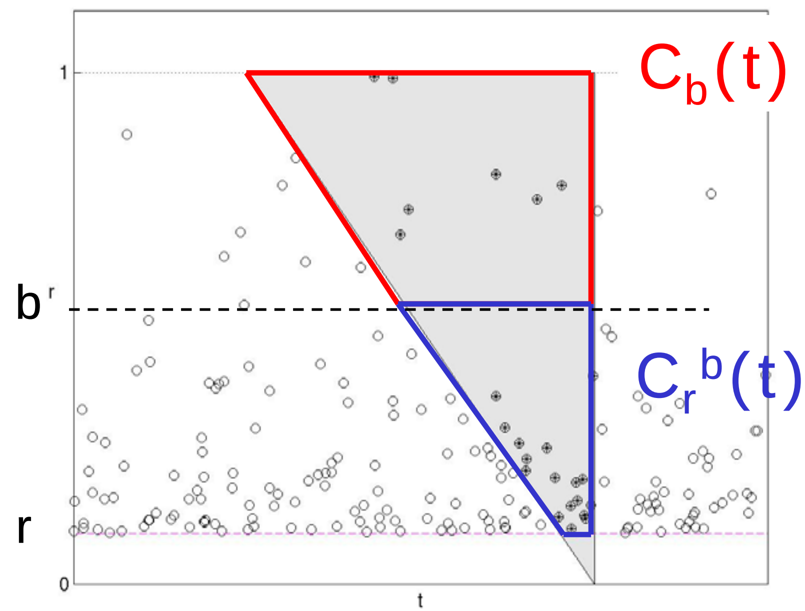
$$dm(t,r) = g(r)dtdr$$

Infinitely divisible nature

Exploit affinity of cone:

$$\mathcal{C}_r^b = \{(t_i, r_i) \in \mathcal{C}_r \,:\, r < b\} \qquad \mathcal{C}_r = \mathcal{C}_r^b \cup \mathcal{C}_b$$

$$\begin{aligned} Q_r(t) &= \prod_{C_r(t)} W_i = \prod_{C_b(t)} W_i \times \prod_{C_r^b(t)} W_i \\ &= Q_b(t) \times \underbrace{\prod_{C_r^b} W_i} \\ &= Q_b(t) \times Q_r^b \end{aligned}$$



Note: Q_r is Martingale

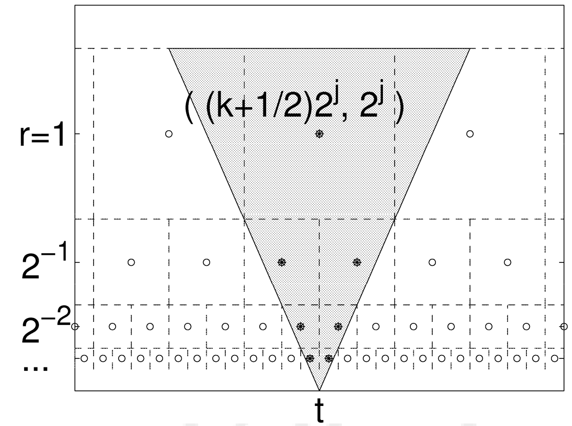
Also:

$$\mathbb{E} Q_r^b(t)^q = \exp\left[-\varphi(q) m(\mathcal{C}_r^b)\right]$$

Binomial: perfect rescaling

- We may write

$$\mu(I_n(\epsilon)) \; = \; \int_{I_n(\epsilon)} 2^n p^{n-l_n}(1-p)^{l_n} dt$$



- The number of multipliers in an octave

$$C_r^b(t) := C_r(t) \backslash C_b(t)$$

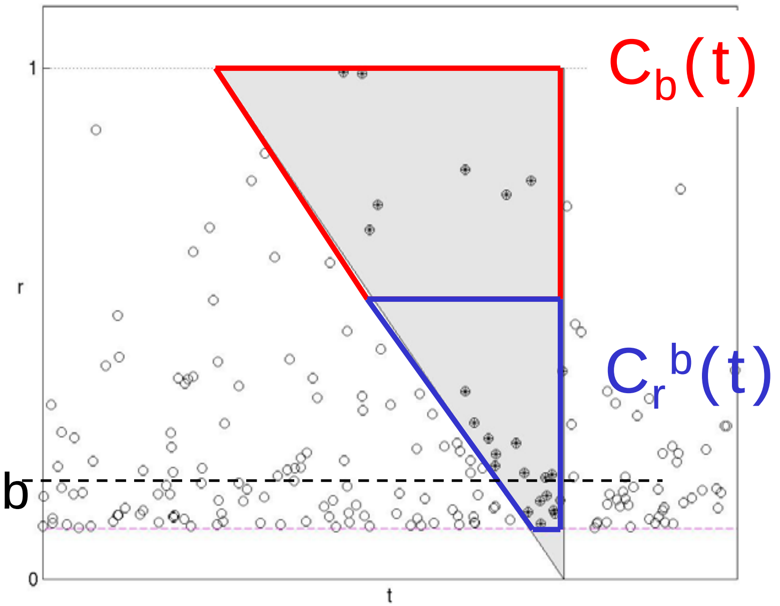
depends only on $b/r \geq 1$.

- Generalize
  - $Q_r^b(t) = \exp[M(C_r^b(t))]$ depends only on b/r (in distribution).
  - $m(C_r^b(t))$ depends only on b/r.
  - $m(C_r^b(t)) = c \log(b/r)$ and $dm(t,r) = c/r^2 dtdr$.

CPC: Invariance and powerlaws

$$\boxed{Q_r(t) \;=\; Q_b(t) \times Q_r^b(t)}$$

$$\boxed{\mathbb{E}Q_r^b(t)^q = \exp\left[-\varphi(q)m(\mathcal{C}_r^b)\right]}$$



Invariance:

- In law:
  Q_r^b rescaled version of $Q_{r/b}$
- $m(\mathcal{C}_r^b)) = m(\mathcal{C}_{r/b})$
- $m(\mathcal{C}_r^b)) = c\log(b/r)$
- $dm(t,r) = c/r^2 dtdr$
- powerlaws: $$Q_{b^n}(t) \;=\; Q_b(t) \times Q_{b^2}^b(t) \ldots \times Q_{b^n}^{b^{n-1}}(t)$$

$$\mathbb{E}Q_r(t)^q = \exp\left[-\varphi(q)c\log(r)\right] = r^{-c\varphi(q)}$$

Scaling: beyond powerlaws

- Degrees of freedom:
  - Shape of Cone can be compensated by transforming the scale $(t,r) \rightarrow (t,r')$
  - Corresponds to warped control measure
- Interest in rich behavior at small scale:
  - Control measure “explodes” at $r=0$
- Scale-invariance goes along with powerlaws

Credits

- Continuous multiplicative cascades from stochastic equations Schmitt [Marsan 2001]
- Multifractal random walk [Bacry Delour Muzy 2001]
- Products of pulses Barral [Mandelbrot 2002]
- Log-infinitely divisible cascades [Bacry Muzy 2002]
- Compound Poisson Cascades, Infinitely divisible cascades [Chainais Abry R. 2002; 2003]

Infinitely Divisible Cascades

Advanced Properties:

Convergence and non-degeneracy

Degenerate cascades

- $\{Q_r(t)\}_{r>0}$: positive, left-continuous martingale
  - Converges almost surely
- Q degenerates
  - As $r \to 0$, $Q_r(t) \to 0$ for a.a. t, a.s.
  - Reason: LLN for log[Q] and E[log Q]<log E[Q]=0
- Set

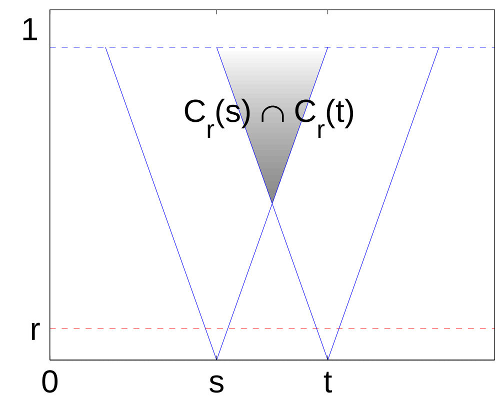
$$A_r(t) = \int_0^t Q_r(s)ds$$

- $\{A_r(t)\}_{r>0}$ is positive martingale
- Non-degenerate:

  If A converges in L_p (p>1) then

$$\mathbb{E}[A(t)] = \lim_r \mathbb{E}[A_r(t)] = \lim_r \int_0^t \mathbb{E}[Q_r(s)]ds = t$$

L2 convergence



- L2 convergence and correlations are governed by control measure of cone intersections, ie, of cones

- A converges in L2 iff

$$\mathbb{E}[A_r(t)^2] = \int_0^t \int_0^t \mathbb{E}[Q_r(u)Q_r(v)]\,du\,dv < K.$$

$$\mathbb{E}[Q_r(u)Q_r(v)] = \exp\left\{-\varphi(2) m\left(\mathcal{C}_r(u) \cap \mathcal{C}_r(v)\right)\right\}$$

- Scale invariant m:

$$\mathbb{E}[Q_r(t)Q_r(s)] = \begin{cases} |t-s|^{c\varphi(2)}\, e^{-c\varphi(2)(|t-s|-1)} & \text{for } r \le |t-s| \le 1 \\ 1 & \text{for } 1 \le |t-s|. \end{cases}$$

Convergence in Lq

Let $1 < q \leq 2$. *A sufficient condition for convergence of* $A_r(t)$ *in* $\mathcal{L}^q$ *is*

$$\limsup_{n\to\infty} \frac{1}{n} m(C_{tk_o^{-n-1}}) < \frac{q-1}{\varphi(q)} \log(1/k_o) \qquad (1)$$

for some integer $k_o \geq 2$ *(recall that* $\varphi(q) < 0$*). In the scale invariant case of* $m(C_r) = -c \log(r)$ *this becomes*

$$(q-1) + c\varphi(q) > 0. \qquad (2)$$

[Barral] for CPC, extension to IDC [CAR]

Infinitely Divisible Cascades

Scaling

Parameter estimation

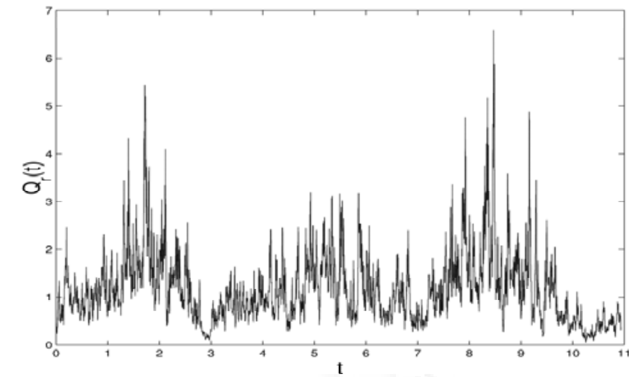
Estimation issues

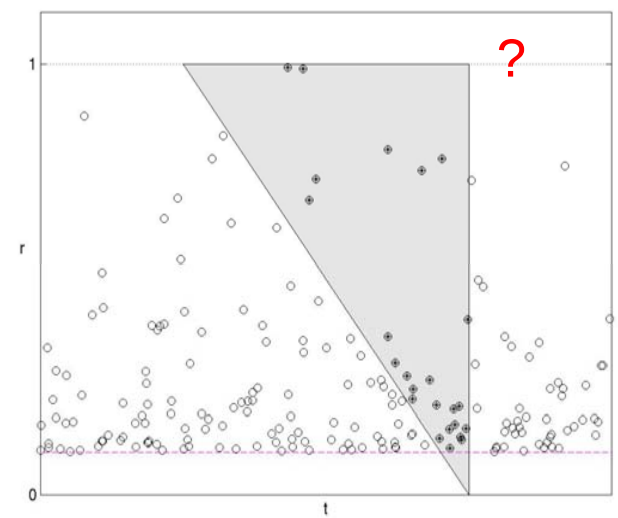
- In practice we observe only Q_r and not the entire cascade
- To recover hidden scaling information consider

$$A_r(t) = \int_0^t Q_r(s)\,ds$$

- ...and study
  - A.s. limit $\quad A(t) = \lim_{r \to 0} A_r(t)$
  - Scaling of limit

$$\mathbb{E}[A(t)^q] = \mathbb{E}[(A(t+s) - A(s))^q]$$





Auto-correlation

- Recall

$$\mathbb{E}[Q_r(u)Q_r(v)] = \exp\{-\varphi(2)m(\mathcal{C}_r(u) \cap \mathcal{C}_r(v))\}$$

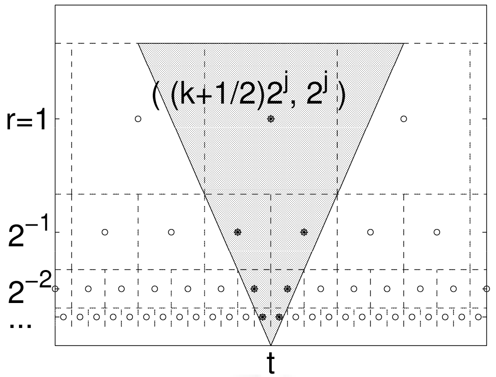
→ Estimation of dm=g(r)dtdr

- Examples:
  - Poisson $\varphi(2) = 1 - (1-e)^2$
  - Compound Poisson $\varphi(2) = -\mathbb{E}[(W-1)^2]$
  - Gaussian $\varphi(2) = -\sigma^2$
  - Stable $\varphi(2) = -\sigma^\alpha(2-2^\alpha)\left(1 - \tan(\frac{\pi\alpha}{2})\right)$

Binomial: scaling of moments

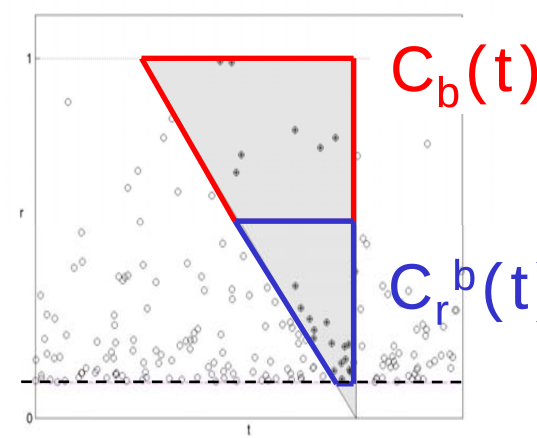
- We may write

$$\mu(I_n(\epsilon)) \;=\; \int_{I_n(\epsilon)} 2^n p^{n-l_n}(1-p)^{l_n} dt$$



- The density “Q” is constant over dyadic intervals
  - Convergence straightforward
  - Moments scale perfectly…
  - …at least over perfectly dyadic scales
- Generalize

$$A_r(t) = \int_0^t Q_b(s) Q_r^b(s) ds = b \int_0^t Q_b(s) d \left[\underbrace{A_{r/b}^{(1)}\left(\frac{s}{b}\right)} \right].$$



IDC independent of Q_b

Scaling of A(t)

Fix $q > 0$, $b \in (0,1)$, $\rho(\cdot)$ *and* $dm = g(r)dtdr$.

- *Moment condition:* A_r *converges in* $\mathcal{L}^q$.
- *Variational condition: "Technical"*
e.g.: CPC and log-normal
- *Speed condition: "sub-invariant",*
e.g., $dm(t,r) = r^\beta dtdr$ ($\beta \le 2$) *and log-deviations.*

Then there exist constants $\overline{C}_q$ *and* $\underline{C}_q$ *such that for any* $t < 1$

$$\underline{C}_q t^q \exp\left[-\varphi(q)m(\mathcal{C}_t)\right] \le \mathbb{E}A(t)^q \le \overline{C}_q t^q \exp\left[-\varphi(q)m(\mathcal{C}_t)\right].$$

Convergence in Lp

- Under assumptions of “scaling” theorem

- Scale-invariant case ($dm(t,r)=c/r^2\, dtdr$)
  - Sufficient condition for convergence in Lp
  $$(p-1)+c\varphi(p)>0$$
  - Necessary condition for convergence in Lp
  $$(p-1)+c\varphi(p)\geq 0$$

Proof idea

$$\mathbb{E}[A(t)^p] \geq 2\mathbb{E}[A(t/2)^p] \simeq 2(t/2)^p(t/2)^{c\varphi(p)}$$

Summary IDC scaling

- Multifractal formalism holds in self-similar case [Barral-Mandelbrot]
- Infinitely Divisible Scaling

Recall

$$\mathbb{E}Q_r(t)^q = \exp\left[-\varphi(q)m(\mathcal{C}(r,0))\right]$$

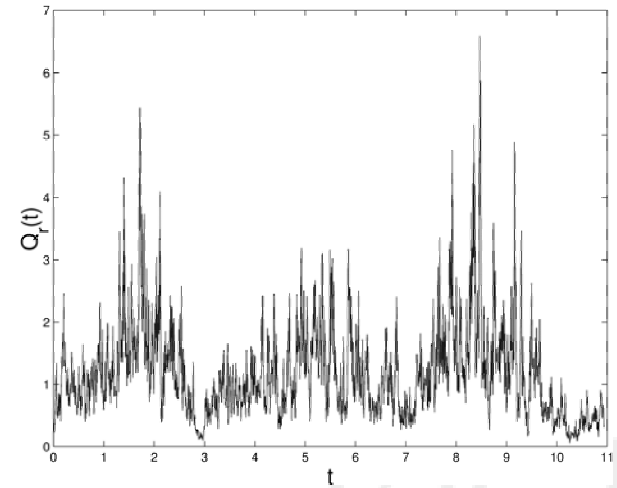
$$\boxed{\mathbb{E}A(t)^q \simeq t^q \exp\left[-\varphi(q)m(\mathcal{C}(t,0))\right]}$$

- powerlaw only if m(C(r,0)) = -c log(r)
- Established
  - for IDC in self-similar case [Bacry-Muzy,Barral]
  - for CPC and log-normal IDC in certain non-powerlaw cases [Chainais-R-Abry]

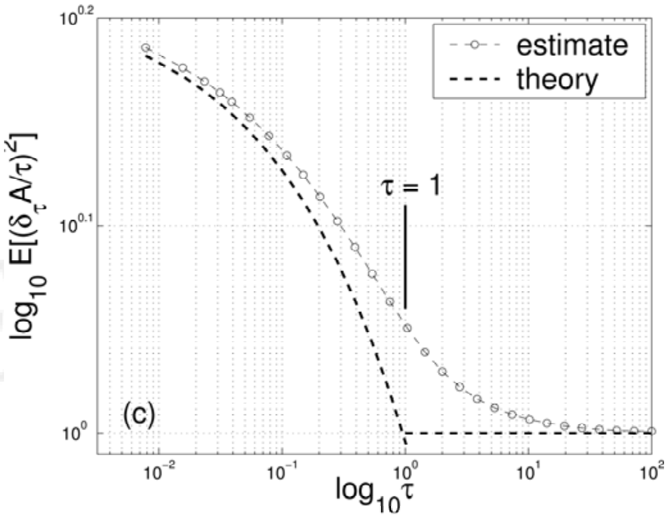
Simulation and Estimation

Simulations of CPC

- Stationary Cascade:



- Non-powerlaw scaling

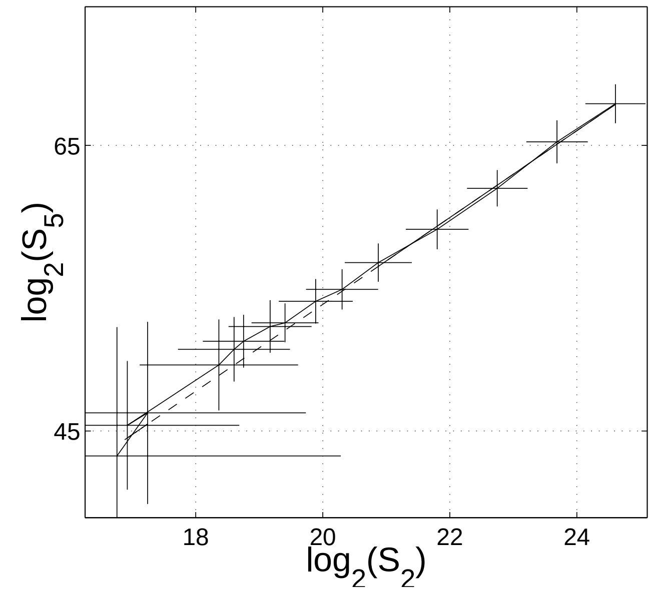


Empirical analysis of network traffic

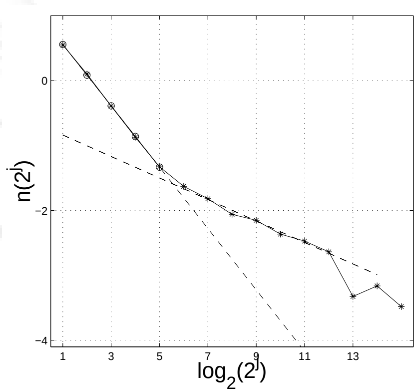
IDC scaling:

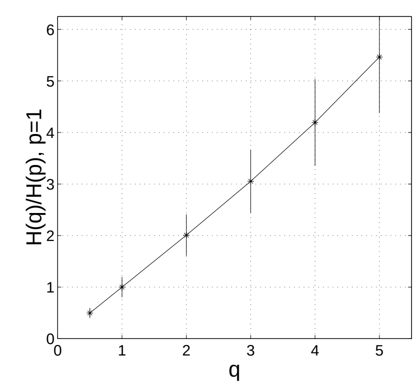
$$
\begin{aligned}
S_q &:= \mathbb{E}[|X(t+\delta) - X(t)|^q] \\
H(q) &:= \log S_q \simeq n(\delta)\zeta(q) \\
\text{fix } p&: \\
H(q)/H(p) &\simeq \zeta(q)/\zeta(p)
\end{aligned}
$$

independent of scale



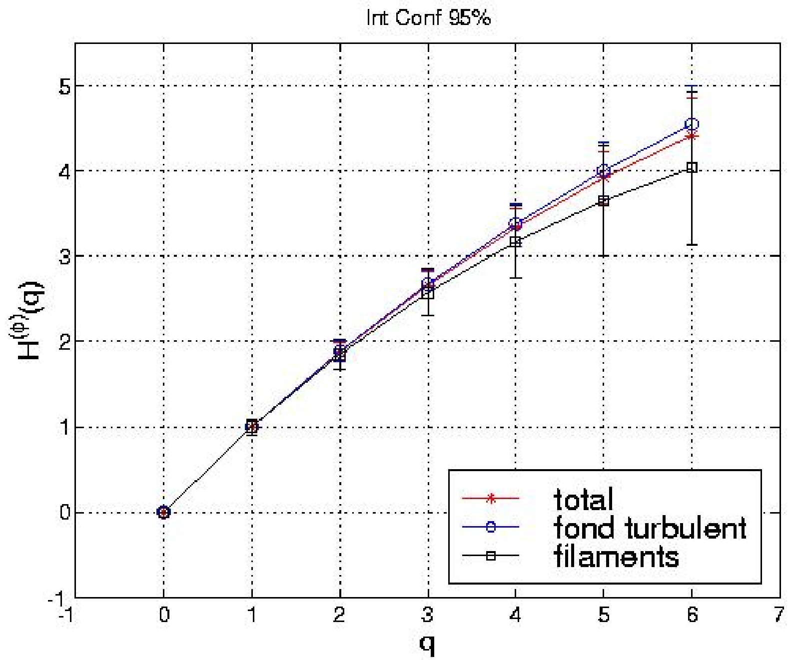
- Clear indication of transition in speed
- Different mechanisms at work at small and large scales

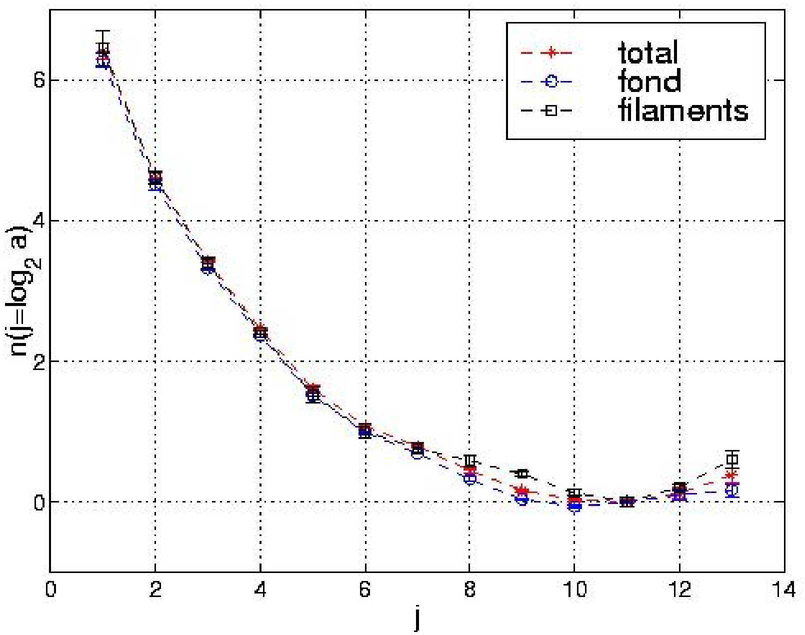




Experimental results (turbulence)

Courtesy P. Chainais





H(q)

n(a): non-powerlaw

Multifractal Subordination

Processes with multifractal oscillations

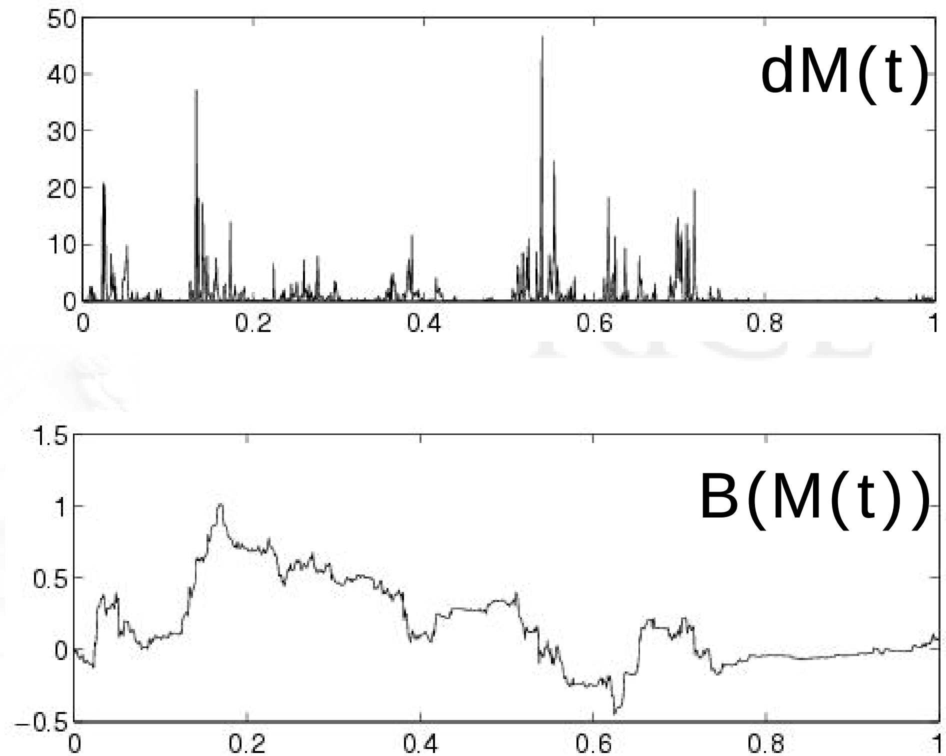
Multifractal time warp

$B_H(M(t))$: B_H fBm, dM independent measure

A versatile model

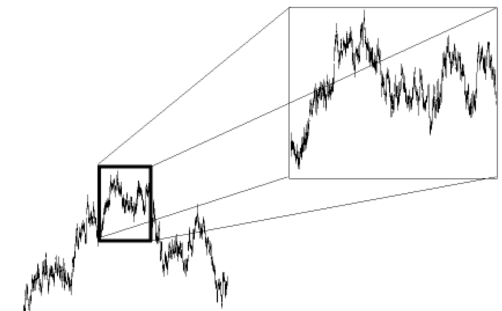
- $M(t)$: Multifractal
  Time change
  Trading time

- B: Brownian motion
  Gaussian fluctuations



Hölder regularity

- Levy modulus of continuity:
  - With probability one for all t



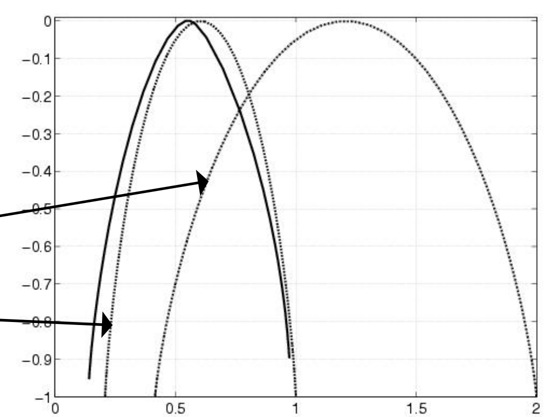
$$|B_H(t+\delta) - B_H(t)| \simeq |\delta|^H$$

  - Thus, exponent gets stretched:

$$|B_H(M(t{+}\delta)){-}B_H(M(t))| \simeq |M(t{+}\delta){-}M(t)|^H \simeq |\delta|^{H\alpha(t)}$$

  - and spectrum gets squeezed:

$$\text{dim}E_a[B_H(M)] = \text{dim}E_{a/H}[M]$$

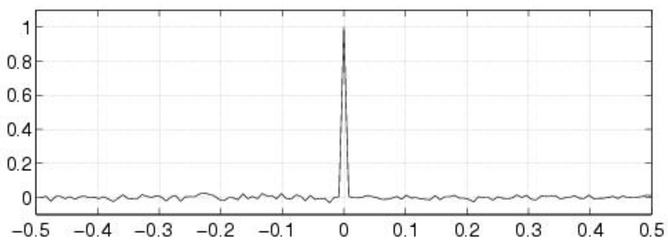


Multifractal Estimation for B(M(t))

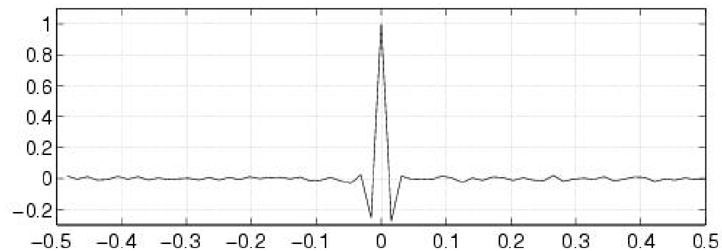
- Weak Correlations of Wavelet-Coefficients:

  (with P. Goncalves)

Haar



Daubechies2



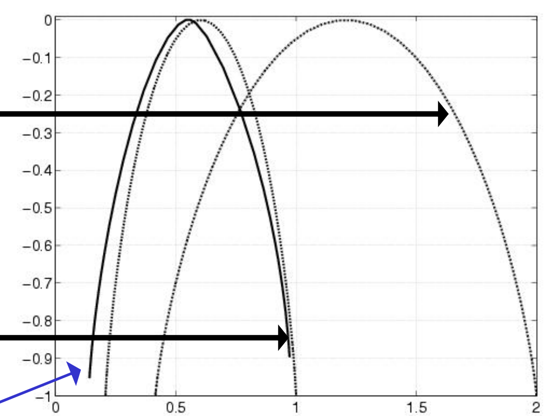
- Improved estimator due to weak correlations
- Multifractal Spectrum

  $M(t+s) - M(t) \sim s^{a(t)}$

  $B(t+u) - B(t) \sim u^{H} \quad (\forall\, t)$

  $\rightarrow$

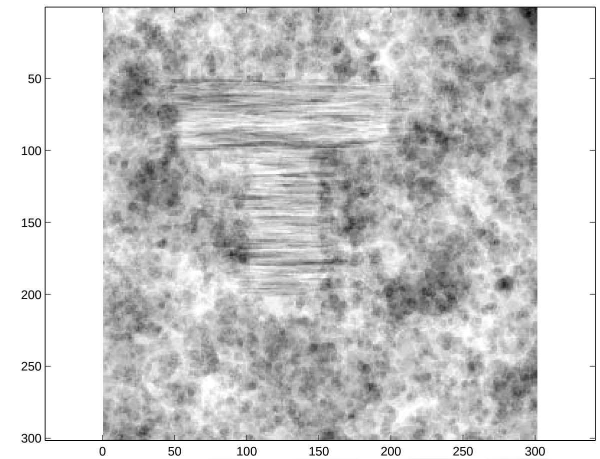
  $B(M(t+s)) - B(M(t)) \sim s^{H*a(t)}$

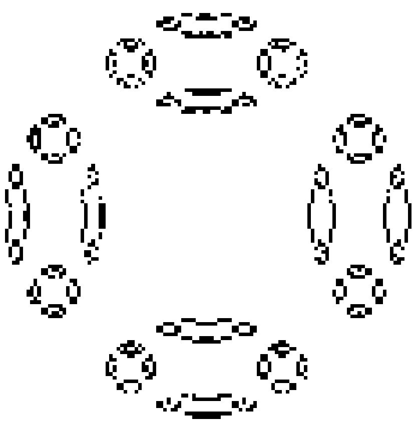


Estimation

Take home

- IDC:
  - Stationary increments
  - Continuous multiplication
  - Versatile scaling
- Unexplored
  - Higher dimensions: anisotropy
  - LRD
  - Pulse shapes
  - Estimation Theory





Reading on this talk

- www.stat.rice.edu/~riedi
- This talk
- Intro for the “untouched mind”
  - Explicit computations on Binomial
- Monograph on “Multifractal processes”
  - Multifractal formalism (proofs, references)
  - Multifractal subordination (warping)
- Papers on “Infinitely Divisible Cascades” [with Chainais and Abry]
- links

The end