From Cascades to Multifractal Processes

Rolf Riedi

Dept of Statistics

WAMA2004, Cargese, July 2004
Reading on this talk

- www.stat.rice.edu/~riedi
- This talk
- Intro for the “untouched mind”
  - Explicit computations on Binomial
- Monograph on “Multifractal processes”
  - Multifractal formalism (proofs, references)
  - Multifractal subordination (warping)
- Papers, links
Why Cascades

Turbulence: models wanted

- Kolmogorov 1941:
  \[< [v(x+r)-v(x)]^q > \sim r^{q/3} \]
- Kolmogorov 1962:
  \[< [v(x+r)-v(x)]^q > \sim r^{H(q)} \]
- ...and beyond

Courtesy P. Chainais
Measured Data

- Networks
- Geophysics
- WWW
- Stock Markets
Multifractal Analysis

Toy Example
The Toy: Binomial Cascade

- Start with unit mass
- Redistribute uniformly portion $p < \frac{1}{2}$ to the left portion $1-p$ to the right
- Iterate
- Converges to measure $\mu$

$$t = \sum_{k=1}^{\infty} \varepsilon_k/2^k \quad \text{with} \quad \varepsilon_k = 0, 1$$

$$I(\varepsilon_1 \ldots \varepsilon_n) := \left[ \sum_{k=1}^{n} \varepsilon_k/2^k, \sum_{k=1}^{n} \varepsilon_k/2^k + 1/2^n \right)$$

$$l_n(t) := \# \{k < n : \varepsilon_k = 1 \} = \sum_{k=1}^{n} \varepsilon_k$$

$$\mu(I(\varepsilon_1 \ldots \varepsilon_n)) = p^{n-l_n(t)}(1-p)^{l_n(t)}$$
Multifractal Spectrum

- Oscillate $\sim |t|^\alpha \rightarrow$ local strength $\alpha$
  $$\alpha(t) := \liminf_n \alpha_n(t)$$
  $$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$I_n(t)$: dyadic interval containing $t$

$\Delta I_n(t)$: oscillation indicator
  total increment over $I_n$,
  max increment in $I_n$,
  wavelet coefficients,…

- Collect points $t$ with same $\alpha$:
  $$E_{\alpha} := \{t : \alpha(t) = a\}$$

- $\text{Dim}(E_{\alpha})$: Spectrum
  $\rightarrow$ prevalence of $\alpha$
Binomial

We take dyadic partition:

\[ I_n(t) = I(\epsilon_1 \ldots \epsilon_n) := \left[ \sum_{k=1}^{n} \epsilon_k / 2^k, \sum_{k=1}^{n} \epsilon_k / 2^k + 1 / 2^n \right) \]

\[ \Delta I_n(t) = \mu(I_n(t)) = p^{l_n(t)} (1 - p)^{n - l_n(t)} \]

\[ \alpha_n(t) = -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1 - p) \]

Range of exponents:

\[ t = 0: \ l_n = 0, \ \alpha_n \to a_\infty := -\log_2(p) < 1 \]

\[ t = 1: \ l_n = n, \ \alpha_n \to a_{-\infty} := -\log_2(1 - p) > 1 \]
“Typical” exponents

t=0, t=1 seem “atypical”.
Intuition: for a “typical” t:

\[ l_n(t) \simeq n/2 \]

Rigorously: Law of Large Numbers

- Binary digits \( \epsilon_k \) are independent, \( P[\epsilon_k=0] = P[\epsilon_k=1] = \frac{1}{2} \):
- \( t \) is uniformly distributed (i.e., with Lebesgue measure \( \mathcal{L} \))

\[
\frac{l_n(t)}{n} = \frac{1}{n} \sum_{k=1}^{n} \epsilon_k \rightarrow \mathbb{E}_{\mathcal{L}}[\epsilon] = 1/2
\]

- “Typical” exponent:

\[
\alpha_n(t) = -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1-p)
\]

\[ \rightarrow a_0 := -\frac{1}{2} \log_2(p) - \frac{1}{2} \log_2(1-p) > 1 \]
A first point on the Spectrum

Conclusion:

• \( \mathcal{L}(E_{a_0}) > 0 \)

• Mass Distribution Principle
  (Lebesgue measure \( \mathcal{L} \) is 1-dim Hausdorff measure)

\[
\dim E_{a_0} = 1
\]

"Where" and "how many" are the other exponents?

• Choose digits "unfairly", e.g., prefer 1 over 0.
Other exponents

The measure $\mu$ prefers 1 over 0 (ratio $1-p$ to $p$).

Intuitive:

$$l_n(t) \simeq n(1 - p)$$

Rigorously: Law of Large Numbers using $\mu$

- Binary digits $\epsilon$ are independent, $P[\epsilon_k=0]=p$, $P[\epsilon_k=1]=1-p$:
- $t$ is distributed according to $\mu$

$$\frac{l_n(t)}{n} = \frac{1}{n} \sum_{k=1}^{n} \epsilon_k \rightarrow \mathbb{E}_\mu[\epsilon] = 1 - p$$

- $\mu$-typical exponent

$$\nu_n(t) = -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1 - p) \rightarrow a_1 := -p \log_2(p) - (1 - p) \log_2(1 - p) < 1$$
A second point on the Spectrum

Conclusion:

• \( \mu(E_{a_1}) > 0 \)

• Mass Distribution Principle \( \rightarrow \) \( \dim E_{a_1} \geq a_1 \)

(Hausdorff dimension of \( \mu \)? It is \( a_1 < 1 \! \)!

\[ \alpha_n(t) = \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \]

• All exponents: Inspiration from Large Deviation Theory
Large Deviations

and the
Multifractal Formalism
Box Spectrum

• Notation:

\[ N_n(a, \delta) := \# \{(\varepsilon_1 \ldots \varepsilon_n) : a - \delta \leq \alpha_n(\varepsilon_1 \ldots \varepsilon_n) < a + \delta\} \]

\[ f(a) := \lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_n \frac{1}{n} \log_2 N_n(a, \delta) \]

• Thm: we always have

\[ \dim E_a \leq f(a) \]

[www.stat.rice.edu/~riedi]

Proof:

Fix \( a \). To prove the first part of the lemma consider an arbitrary \( \gamma > f(a) \), and choose \( \eta > 0 \) such that \( \gamma > f(a) + 2\eta \). Then, there is \( \varepsilon > 0 \) and integer \( m_0 \) such that

\[ N_n(a, \varepsilon) \leq 2^{-(f(a)+\eta)} \]

for all \( n \geq m_0 \). Let us define \( J(m) := \{k_n : n \geq m \text{ and } u - \varepsilon \leq u_k \leq u + \varepsilon\} \). Then, for any \( m \) the intervals \( I_{k_n} \) with \( k_n \in J(m) \) form a cover of \( E_a \). Those intervals are of length less than \( 2^{-m} \). Moreover, for any \( m > m_0 \) we have

\[ \sum_{k_n \in J(m)} |I_{k_n}| \leq \sum_{n=2^m}^{2^{m+1}} N_n(a, \varepsilon) \cdot 2^{-m} \]

\[ < \sum_{n=2^m}^{2^{m+1}} 2^{-m} f(a) \cdot 2^{-m} \]

\[ < \sum_{n=2^m}^{2^{m+1}} 2^{-m} \]

which tends to zero with \( m \to \infty \). Hence, \( \dim E_a \leq \gamma \). Letting \( \gamma \to f(a) \) completes the proof.

[www.stat.rice.edu/~riedi]

• Beware the folklore: \( f(a) \) is NOT the box-dim of \( E_a \)
Legendre spectrum

• Notation: partition sum and function

\[ S_n(q) := \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q = \sum_{\epsilon_1 \ldots \epsilon_n} 2^n |q \alpha_n(\epsilon_1 \ldots \epsilon_n)|. \]

\[ \tau(q) := \liminf_{n \to \infty} \frac{1}{n} \log_2 S_n(q) \]

• Thm: we always have

\[ f(a) \leq \tau^*(a) := \inf_q (qa - \tau(q)) \]

Proof

\[ \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \geq \sum_{\alpha_n(\epsilon_1 \ldots \epsilon_n) \in [a-\delta, a+\delta]} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \geq N_n(a, \delta) 2^{-n(qa+|q|\delta)} \]

\[ \geq 2^{-n(qa-f(a)+\delta'+|q|\delta)} \]
Legendre spectrum

• Thm: provided $\alpha_n(t)$ are bounded we have

$$f(a) = \tau^*(a) \quad \text{for } a = \tau'(q).$$

• Proof idea: steepest ascent (large deviations)

$$\sum_{\varepsilon_1 \ldots \varepsilon_n} |\Delta I_n(\varepsilon_1 \ldots \varepsilon_n)|^q \leq \sum_{i=1}^m \alpha_n(\varepsilon_1 \ldots \varepsilon_n) \sum_{i=i-\delta, i+\delta} |\Delta I_n(\varepsilon_1 \ldots \varepsilon_n)|^q$$

$$\leq \sum_{l=1}^m N_n(l\delta, \delta) 2^{-n(ql\delta - |q|\delta)}$$

$$\leq \sum_{l=1}^m 2^{-n(ql\delta - f(l\delta) - |q|\delta)} \leq m2^{-n(\inf_a (qa - f(a) - |q|\delta))}$$

Thus: $\tau(q) = f^*(q) = \inf_a (qa - f(a))$ for all $q$.

• $\tau$ is concave, non-decreasing, differentiable with exceptions

• Recover $f=f^{**}$ at $a=\tau'(q)$ using lower semi-continuity
Legendre transform 101

- Elementary calculus:
  \[ \tau^*(a) := \inf_q (q a - \tau(q)) = \bar{q} a - \tau(\bar{q}) \]
  where \( \bar{q} \) is defined by \( a = \tau'(\bar{q}) \)
  
  - Tangent of slope \( a \) to \( \tau(q) \)
  - Intersection with ordinate yields \( -\tau^*(a) \)
  - Dual holds
Binomial Spectrum
continued
Partition function of the Binomial

\[ S_n(q) = \sum_{\epsilon_1 \cdots \epsilon_n} |\Delta I_n(\epsilon_1 \cdots \epsilon_n)|^q \]
\[ = \sum_{\epsilon_1 \cdots \epsilon_n} [p^{n-l_n}(\epsilon_1 \cdots \epsilon_n)(1 - p)^{l_n}(\epsilon_1 \cdots \epsilon_n)]^q \]
\[ = \sum_{l=0}^{n} \binom{n}{k} [p^{n-l}(1 - p)^l]^q \]
\[ = [p^q + (1 - p)^q]^n. \]

• (Upper) envelope of \( \text{dim}(E_a) \):

\[ \tau(q) = \liminf_{n \to \infty} \frac{1}{n} \log_2 S_n(q) \]
\[ = -\log_2[p^q + (1 - p)^q] \]
Insight from Large Deviations

- From steepest ascent:

\[ S_n(q) = \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \simeq 2^{-n(\inf_a (qa-f(a))} \]

\[ = 2^{-n(q\bar{a}-f(\bar{a}))} \simeq \sum_{\alpha_n(\epsilon_1 \ldots \epsilon_n) \simeq a} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \]

- Dominant terms in \( S_n(q) \), for fixed \( q \), are the ones with

\[ \alpha_n(\epsilon_1 \ldots \epsilon_n) = \frac{\log \Delta I_n}{\log |I_n|} \simeq \bar{a} = \tau'(q) \]

- ...and vice versa: these terms contribute such that

\[ S_n(q) \simeq 2^{-n\tau(q)} = (p^q + (1-p)^q)^n \]

For the Binomial these correspond to mass re-distribution in ratio \( p^q \) to \( (1-p)^q \)
Locating the exponents

Fix $q$.

Consider the measure $\mu_q$ defined as $\mu$ but with mass ratio $pq$ to $(1-p)^q$. Intuitively, we have then:

$$l_n(t) = \#\{k \leq n : \epsilon_k = 1\} \simeq n \frac{(1-p)^q}{pq + (1-p)^q} = n(1-p)^q 2^{\tau(q)}$$

Rigorously: Law of Large Numbers using $\mu_q$

- Binary digits $\epsilon$: indep, $P[\epsilon_k=0]=pq2^{\tau(q)}$, $P[\epsilon_k=1]=(1-p)^q2^{\tau(q)}$
- $t$ is distributed according to $\mu_q$

$$\frac{l_n(t)}{n} = \frac{1}{n} \sum_{k=1}^{n} \epsilon_k \rightarrow \mathbb{E}_{\mu_q}[\epsilon] = (1-p)^q 2^{\tau(q)}$$

- $\mu$-typical exponent

$$\alpha_n(t) = -\frac{n - l_n(t)}{n} \log_2(p) - \frac{l_n(t)}{n} \log_2(1-p)$$

$$\rightarrow a_q := -p^q 2^{\tau(q)} \log_2(p) - (1-p)^q 2^{\tau(q)} \log_2(1-p)$$
Completing the Spectrum

Conclusion:

- \( \mu_q(E_{a_q}) > 0 \)
- \( a_q = \tau'(q) \)

- Hausdorff dimension of \( \mu_q \):

\[
\frac{\log \mu_q(I_n(t))}{\log |I_n(t)|} = -\frac{n - l_n(t)}{n} \log_2[p^q 2^{\tau(q)}] - \frac{l_n(t)}{n} \log_2[(1 - p)^q 2^{\tau(q)}]
\]

\[
= -\tau(q) + q \alpha_n(l)
\]

\[
\rightarrow qa_q - \tau(q) = \tau^*(a_q)
\]

- Mass Distribution Principle

\[
\dim E_{a_q} \geq \tau^*(a_q)
\]
Lessons

Binomial cascade: \( \dim E_\alpha = f(\alpha) = \tau^*(\alpha) \)

- Points with exponent \( \log_\mu(I(t))/\log|I(t)| \sim a = \tau'(q) \)
  - Are concentrated on the support of \( \mu_q \)
  - Dominate the partition sum \( S_n(q) \)
- Partition function allows to bound/estimate \( \dim(\mathcal{E}_\alpha) \)
Random Cascades

A further multifractal envelop
Convergence and Degeneracy
Multifractal Spectra and Randomness

- $\Delta I_n(t)$: oscillation indicator for process or measure
- Pathwise
  \[ \dim E_a \leq f(a) \leq \tau^*(a) \]
- $S_n(q)$ is $q$-th moment estimator.
- Replace by true moment:
  \[ T(q) := \liminf_{n \to \infty} \frac{1}{n} \log_2 \mathbb{E} S_n(q) \]

...analytically easier to handle and often sufficient
- $T(q)$ is concave like $\tau(q)$, but NOT always increasing
Pathwise and deterministic envelop

• Lemma: With probability one for all $q$ with $T(q) < \infty$.
  \[ \tau(q, \omega) \geq T(q) \]
  [Proof: www.stat.rice.edu/~riedi]

• Cor: \[ \tau^*(a, \omega) \leq T^*(a) \]

• Weaker result from Chebichev inequality:
  \[ \mathbb{E} \left[ \log(X) \right] \leq \log \mathbb{E}[X] \]

• Material science: free energy is “self-averaging” iff quenched and annealed averages are equal.
Multifractal Envelopes

- Almost surely, for all $a$:
  \[ \dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a) \]

- Holds \textit{always} provided use same $\Delta I_n$ in all spectra

- Choice of scales $I_n$
  - $I_n$ is here \textit{dyadic}, could be any sub-exponential
  - This could affect/change $f$, $\tau$ and/or $T$ due to boundary effects
  - Robust: $\Delta I_n = \text{oscillation in } I_n \text{ and its neighbor intervals}$

- Choice of \textit{oscillation indicator} $\Delta I_n$
  - For true Hoelder regularity $\Delta I_n = \max \text{ increment "around" } I_n$
  - $\Delta I_n = \text{Wavelet coefficient: only a proxy to Hoelder regularity!}$
  - For measures supported on $[0,1]$: $\Delta I_n = \mu(I_n)$ gives Hoelder!
Multifractal Envelops

- Almost surely, for all $a$:
  $$\dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a)$$

- Special feature:
  - If a property of "bounded total variation" holds then the spectrum $f$ touches the bi-sector:
    If $\sum_{\epsilon_1, \ldots, \epsilon_n} \Delta I_n(\epsilon_1 \ldots \epsilon_n) \leq C$ for all $n$ then $\tau(1) = 0$. 

Recall at $a = \tau'(q)$
$$f(a) = \tau^*(a)$$
Multifractal Envelops

- Almost surely, for all $a$:
  \[ \dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a) \]

- Terminology:
  - Multifractal formalism “holds” if
    \[ \dim(E_a) = f(a) = \tau^*(a) \]
    with your preferred oscillation indicators $\Delta I_n$, e.g., Holder exponent in $E_a$, wavelet decay in $f(a)$.
    [First step: show $T$ is same for Holder and wavelets.]

  - Falconcer: “A concise definition of a multifractal tends to be avoided.”
  - Others: “An object is multifractal if the formalism holds for it.”
  - Others: “An object is multifractal if it has more than one singularity exponent”. (not mono-fractal)
Multifractals
and classical regularity
Besov spaces

• For oscillation indicator from wavelets:

\[
\sup \{ s : Y \in B_v^s(L_u) \} = \frac{\tau(u) + 1}{u}
\]

• Proof: use wavelet coefficients \( C_{j,k} = \Delta I_j(2^j k) \) and equivalent Besov norm

\[
\left( \sum_k |D_{0,0}|^v \right)^{1/v} + \left( \sum_{j > J_0} \left( \sum_k 2^{jsu} 2^{-j} |2^j C_{j,k}|^u \right)^{v/u} \right)^{1/v} .
\]
Kolmogorov

• Thm [Kolmogorov]:
  – If \( E[ |A(s)-A(t)|^b] < C |s-t|^{1+d} \) then almost all paths of \( A \) are of (global) Holder-continuity for all \( h < d/b \),
  – i.e., for all \( h < T(q)/q \).

• The best such \( h \) is \( \min(a : T^*(a)>0) \).
  – \( T(q)/q = \) slope of tangent through the origin.
Binomial Spectrum

continued
Binomial with Random Multipliers

- Random re-distribution
- Multipliers Independent between scales

\[ \mu_n(I(\epsilon_1 \ldots \epsilon_n)) = M\epsilon_1 \ldots M\epsilon_1 \ldots \epsilon_n \]

- Conservative:

\[
M\epsilon_1 \ldots \epsilon_n 0 + M\epsilon_1 \ldots \epsilon_n 1 = 1
\]

- Conservation is too restrictive for stationarity!

"Martingale de Mandelbrot":

\[
\mathbb{E}[M\epsilon_1 \ldots \epsilon_n 0 + M\epsilon_1 \ldots \epsilon_n 1] = 1
\]
Convergence of Random Binomial

• Conservative:
  
  - $M_{\epsilon_1...\epsilon_n 0} + M_{\epsilon_1...\epsilon_n 1} = 1$
  
  - For all $m > n$
    
    $\mu_m(I(\epsilon_1...\epsilon_n)) = M_{\epsilon_1}...M_{\epsilon_1...\epsilon_n}$
  
  - Thus converges to
    
    $\mu(I(\epsilon_1...\epsilon_n)) = M_{\epsilon_1}...M_{\epsilon_1...\epsilon_n}$
Convergence of Random Binomial

- “Martingale de Mandelbrot”:
  - A price to pay towards stationarity
  - $\mathbb{E}[M_{\epsilon_1...\epsilon_{n_0}} + M_{\epsilon_1...\epsilon_{n_1}}] = 1$
  - Martingale: For all $m > n$
  - $\mathbb{E}[\mu_m(I(\epsilon_1 \ldots \epsilon_n))|\mathcal{F}_n] = M_{\epsilon_1} \ldots M_{\epsilon_1 \ldots \epsilon_n} = \mu_n(I(\epsilon_1 \ldots \epsilon_n))$
  - Thus converges almost surely (but may degenerate)
  - We have
    $\mathbb{E}[\mu(I(\epsilon_1 \ldots \epsilon_n))|\mathcal{F}_n] = M_{\epsilon_1} \ldots M_{\epsilon_1 \ldots \epsilon_n}$
Envelope for Random Binomial

- By independence of multipliers
  - Martingale of Mandelbrot:

\[
\mathbb{E}[S_n(q)] = \sum_{\epsilon_1 \ldots \epsilon_n} \mathbb{E} |\Delta l_n(\epsilon_1 \ldots \epsilon_n)|^q = \sum_{\epsilon_1 \ldots \epsilon_n} \mathbb{E} |M_{\epsilon_1} \ldots M_{\epsilon_1 \ldots \epsilon_n}|^q = 2^n \mathbb{E}[M^q]^n.
\]

\[
T(q) = -1 - \log_2 \mathbb{E}[M^q]
\]

- Conservative: similar

\[
\mathbb{E}[S_n(q)] = \sum_{\epsilon_1 \ldots \epsilon_n} \mathbb{E}[M^q]^{n-l_n(\epsilon_1 \ldots \epsilon_n)} \mathbb{E}[(1 - M)^q]^{l_n(\epsilon_1 \ldots \epsilon_n)}
\]
\[
= (\mathbb{E}[M^q] + \mathbb{E}[(1 - M)^q])^n
\]
\[
= (2\mathbb{E}[M^q])^n.
\]
Kahane-Peyriere theory for the Martingale of Mandelbrot

• Martingale “degenerates”
  – iff $\mu([0,1])=0$ almost surely zero
  – iff $E\mu([0,1])=0$
  – iff $T'(1)\leq0$

• Intuition:
  – $T'(1) = a_1 = \text{dimension of the carrier of } \mu$
  – If $T'(1)>0$ then
    • $\exists q>1$ with $T(q)>0$
    • $\mu$ converges in $L_q$
    • $\mathbb{E}[\mu([0,1])] = \lim_n \mathbb{E}[\mu_n([0,1])] = 1$
Multifractal formalism holds

- Thm for random **binomial** [Barral, Arbeiter-Patschke, Falconer]:
  - Set $\Delta I_n = \mu(I_n)$.
  - Assume $M$ has a **finite** moment of some **negative** order
  - Then, with probability $1$: for all $a$ such that $T^*(a)>0$

\[
\dim E_a = f(a) = \tau^*(a) = T^*(a)
\]

- **Note:**
  - $T^*(a)>0$ means $a=T'(q)$ with $q$ limited by tangents through the origin: $T'(q)=T(q)/q$.
  - Little known in general for other $a$ ...or $q$! Possible: $\tau(q)>T(q)$
  - Proofs: Use Mass distortion Principle with factors $M^q$
Wavelets for the Binomial

- Compactly supported wavelet
  - $\Delta I_n =$ wavelet coefficient corresponding to $I_n$
  - $\Delta I_n$ same rescaling property as measure itself
  - Same $T(q)$
  - Multifractal formalism holds
Toy examples

White noise

Cascade
Log-Normal Binomial

- Deterministic envelope is a parabola: [Mandelbrot]

\[ T(q) = (q-1) \left(1 - \frac{\sigma^2}{2 \ln(2)} q \right) \quad \text{for} \quad q < q_{\text{crit}} := \frac{2 \ln(2)}{\sigma^2}. \]

- Zeros: \( q=1, \ q=q_{\text{crit}} \)

- Non-Degeneracy:
  \[ T'(1) > 0 \iff q_{\text{crit}} > 1 \iff 2 \ln(2) > \sigma^2 \]

- Spectrum is parabola as well
  \[ T^*(\omega) = 1 - \frac{\ln(2)}{2\sigma^2} \left( \omega - 1 - \frac{\sigma^2}{2 \ln(2)} \right)^2 \]

- Partition function \( \tau(q) \) is non-decreasing,
- thus \( \tau(q) > T(q) \) (at least) for \( q > (1+q_{\text{crit}})/2 \)
Multifractal Product of Pulses

together with
I. Norros and P. Mannersalo
Network Traffic is Multifractal

- Visually striking
- Scaling of impressive quality
  (Levy Vehel & RR ’96,
  Norros & Mannersalo ’97,
  Willinger et al ’98)
- Statistical models:
  - Binomial cascades with
    scale dependent multipliers
    (Crouse & RR ’98, Willinger et al ’98)
- Not stationary!
  - Cumbersome for statistics
  - and probability (Queueing)
Multifractal paradigm

Multiplicative Processes:
– From redistributing mass to multiplying pulses

\[ A(t) = \lim_{n \to \infty} \int_0^t \Lambda_0(s) \ldots \Lambda_n(s) \, ds \]

Binomial Cascade
– \( \Lambda_n(s) \) is constant on dyadic intervals

– Conservative:
\( \Lambda_n(2k/2^n) + \Lambda_n((2k+1)/2^n) = 2 \)

– Martingale de Mandelbrot:
\( E \Lambda_n(s) = 1 \)

– Not stationary
Multifractal paradigm

• Multiplicative Processes:

\[ A(t) = \lim_{n \to \infty} \int_{0}^{t} \Lambda_0(s) \cdots \Lambda_n(s) \, ds \]

• Stationary Cascade
  - \( \Lambda_n(s) \) is stationary
  - Conservation:
    \[ E\Lambda_n(t) = 1 \]
  - "self-similarity":
    \[ \Lambda_n(s) = d \Lambda_1(sb^n) \]
Parameters and Scaling

• Parameter estimation
  \( \Lambda_i(s) \): i.i.d. values with Poisson arrivals (\( \lambda_i \)):
  \[ Z(s) = \log \left[ \Lambda_1(s) \Lambda_2(s) \ldots \Lambda_n(s) \right] \]
  \[ \text{Cov}(Z(t)Z(t+s)) = \sum_{i=1..n} \exp(-\lambda_i s) \text{Var} \Lambda_i(s) \]

• Performance of predictors / simulations

\[ \Lambda_1 \Lambda_2 \ldots \Lambda_n \]

• Multifractal Envelope
  (with Norros and Mannersalo)
  \[ T(q) = q - 1 - \log_2 \mathbb{E}[\Lambda^q] \]
Interlude

Self-similar processes
Statistical Self-similarity

• Self-similarity: canonical form
  – $B(at) = \overset{\text{fdd}}{=} C(a) B(t)$ \quad B: process, C: scale function
  – Iterate: $B(abt) = \overset{\text{fdd}}{=} C(a)C(b) B(t)$
  – $C(a)C(b) = C(ab)$
  \rightarrow C(a) = a^H : \text{Powerlaw is default}

• $H$-self-similar:

\[ B(at) = \overset{\text{fdd}}{=} a^H B(t) \]

• Examples
  – Gaussian: fractional Brownian motion $B_H(t)$ is unique $H$-self-similar Gaussian process with stationary increments.
  – Stable: not unique in general, $a=1/H$: Levy motion
Self-similar Processes

• What do they model?
  – Sustained excursions above/below the mean

• Different from (finite order) linear models
  – Auto-Regressive
  – ARMA
  – (G)ARCH
  – Exponential decay of correlations

• Corresponds to infinite order AR models
  – FARIMA
  – FIGARCH

\[ fBm(t) = \int_{-\infty}^{t} K(t,s) \, dW(s) \]
Multifractal Subordination

Processes with multifractal oscillations
Multifractal time warp

$B_H(M(t))$: $B_H$ fBm, $dM$ independent measure

A versatile model

- $M(t)$: Multifractal
  Time change
  Trading time

- $B$: Brownian motion
  Gaussian fluctuations
Hölder regularity

- Levy modulus of continuity:
  - With probability one for all $t$
    \[ |B_H(t + \delta) - B_H(t)| \simeq |\delta|^H \]
  - Thus, exponent gets stretched:
    \[ |B_H(M(t+\delta)) - B_H(M(t))| \simeq |M(t+\delta) - M(t)|^H \simeq |\delta|^{H\alpha(t)} \]
  - and spectrum gets squeezed:
    \[ \dim E_a[B_H(M)] = \dim E_{a/H}[M] \]
Multifractal formalism for $B_H(M(t))$

- Conditioning on $M$ one finds:

$$
\mathbb{E} |B_H(M(t+\delta)) - B_H(M(t))|^q = \mathbb{E} |B_H(1)|^q \mathbb{E} |M(t+\delta) - M(t)|^{qH} \\
\leq |\delta| T_M(qH)
$$

- thus

$$
T_{B(M)}(q) = T_M(qH)
$$

- which confirms the stretched exponent:

$$
T'_{B(M)}(q) = HT'_M(qH)
$$

- and matches with warp formula before:

$$
T^*_{B(M)}(\alpha) = T^*_M(\alpha/H)
$$

- If the formalism holds for $M$, then also for $B_H(M(t))$
Estimation: Wavelets decorrelate
(with P. Goncalves)

\[ W_{jk} = \int \psi_{jk}(t) B(M(t)) \, dt \]

\( N: \) number of vanishing moments

\[ E[W_{jk} W_{jm}] = \int \int \psi_{jk}(t) \psi_{jm}(s) E[B(M(t)) B(M(s))] \, dt \, ds \]
\[ = \int \int \psi_{jk}(t) \psi_{jm}(s) E[|M(t) - M(s)|^{2H}] \, dt \, ds \]
\[ \sim O( |k-m|^{T(2H) + 1 - 2N} ) \quad ( |k-m| \to \infty ) \]
Multifractal Estimation for $B(M(t))$

- Weak Correlations of Wavelet-Coefficients:
  (with P. Goncalves)

- Improved estimator due to weak correlations
- Multifractal Spectrum

$$M(t+s) - M(t) \sim s^{a(t)}$$

$$B(t+u) - B(t) \sim u^H \quad (\forall t)$$

$$\Rightarrow$$

$$B(M(t+s)) - B(M(t)) \sim s^{H*a(t)}$$

Estimation
From Multiplicative Cascades to Infinitely Divisible Cascades

with

P. Chainais and P. Abry

Independent work: Castaing, Schmidt, Barral-Mandelbrot, Bacry-Muzy
Adapting to the real world

Real world data

• can deviate from powerlaws: traffic
• has no preference for dyadic scales

Lukacs: if the data does not fit to the model then too bad for the data.
Experimental results

H(q)

n(a): non-powerlaw

Courtesy P. Chainais
Beyond Self-similarity

- Self-similarity revisited:
  - $B(at) =^d C(a) B(t)$  \hspace{1cm} B: process, C: scale function
  - $B(abt) =^d C(a)C(b) B(t)$
  - $C(a)C(b) = C(ab) \rightarrow C(a) = a^H$
  - $E[|B(a^n)|^q] = c(q) (a^{qH})^n$
  - linear in $q$ (mono-fractal)
Beyond Self-similarity

• Self-similarity revisited:
  – $B(at) =^d C(a) B(t)$  \( B: \text{process, } C: \text{scale function} \)
  – $B(abt) =^d C(a)C(b) B(t)$
  – $C(a)C(b)=C(ab) \Rightarrow C(a) = a^H$
  – $E[|B(a^n)|^q] = c(q) (a^{qH})^n$
  – linear in $q$ (mono-fractal)

• More flexible rescaling “Ansatz“:
  – $C=C(a,t)$ ? : non-stationary increments
  – $C=\text{independent r.v. for every re-scaling :}$
  – $X(a...at)= X(a^n t) = C_1(a)...C_n(a) X(t)$: \text{multiplicative}
  – $E[|X(a^n)|^q] = c(q) E[|C(a)|^q]^n$
  – non-linear in $q$; \text{powerlaw}
Infinitely divisible scaling

Self-similarity: $\mathbb{E}[|B(t+\delta) - B(t)|^q] \simeq \delta^{qH}$

Multifractal scaling: $\mathbb{E}[|M(t+\delta) - M(t)|^q] \simeq \delta^{1+T(q)}$

IDC scaling: $\mathbb{E}[|X(t+\delta) - X(t)|^q] \simeq \exp[n(\delta)\zeta(q)]$

- Multifractal scaling reduces to self-similarity if $T$ is linear in $q$. (sometimes called mono-fractal)
- IDC reduces to multifractal scaling if $n(\delta)=-\log(\delta)$
- In general $n(\delta)$ gives the speed of the cascade
Geometry of Binomial Pulses

- Time-Scale plane: codes shape of pulses
  - **Position** \( (T=\text{center}) \)
  - **Size** \( (R=\text{length}) \)

Pulses:
\[
P_i(t) = W_i \quad \text{if} \quad |t-t_i|<r_i/2
\]
1 \( \quad \text{else} \)

For Binomial: Strict dyadic geometry
Stationary geometry

Randomize Positions and Sizes

Large Scale Pulses

Medium Scale Pulses
Compound Poisson Cascade

Poisson points \((t_i, r_i)\) in time-scale plane with marks \(W_i\)

Cone of influence at \(t\)

\[ C(r, t) = \{(t_i, r_i) : t - r_i < t_i < t, r_i > r\} \]

Cascade Process:

\[ Q_r(t) = \prod_{(t_i, r_i) \in C(r, t)} W_i \]

- Poisson Cascades exhibit scaling properties akin to IDC scaling

\[
m(C(r, t)) = m(C(r, 0)) = \mathbb{E}[\#\{(t_i, r_i) \in C(r, t)\}]
\]

\[
\mathbb{E} Q_r(t)^q = \exp \left[ -\varphi(q) m(C(r, *)) \right]
\]
Cascade and AR processes

- Continuous version (IDC):
  \[ Q(t) = \exp M(C(t)) = \exp \int k_C(t, s) dM(s) \]
  - M is an infinitely divisible measure

- Classic theory to be exploited:
  - AR-type processes
    \[ B_H(t) = \int \tilde{k}_H(s, t) dW(s) \]
  - kernel estimate of the random measure dM
Cascades: Invariance and scaling

Infinitely divisible nature and scaling of the cascade:

\[ Q_r(t) = \prod_{C(r,t)} W_i = \prod_{C_b} W_i \times \prod_{C_r^b} W_i \]

\[ = Q_b(t) \times \prod_{C_r^b} W_i \]

Rescaled version of \( Q_{r/b} \) in the scale-invariant case only!

Poisson Cascade has re-scaling properties; in scale invariant case: akin to Product of Processes
Multifractal scaling

• Multifractal formalism holds in self-similar case [Barral-Mandelbrot]
• Infinitely Divisible Scaling

Recall
\[ \mathbb{E}Q_r(t)^q = \exp \left[ -\varphi(q)m(C(r, *)) \right]. \]

\[ \mathbb{E}A(t)^q \sim t^q \exp \left[ -\varphi(q)m(C(t, *)) \right] \]

– powerlaw only if \( m(C(t, *)) = -\log(t) \)
– for IDC in self-similar case [Bacry-Muzy, Barral]
– for CPC and log-normal IDC in certain non-powerlaw cases [Chainais-R-Abry]
Simulations

- Stationary Cascade:
- Non-powerlaw scaling
“Never happy”: More flexibility

• Better control of scaling

• Wider range of known non-powerlaw scaling

• Higher dimensions: anisotropy
  – “As expected” in generic cases [Falconer, Olsen]
  – Formalism may break if directional preferences [McMullen, Bedford, Kingman, R]
Overall Lessons

- Multifractal spectrum <-> regularity
  - Besov spaces
  - Global Hölder regularity

- Powerful modeling via multiplication through scales
  - Poisson product of Pulses
  - Multifractal warping
  - Degeneracy: price to pay for stationarity

- Estimation via wavelets
  - Multifractal envelopes
    - numerical $\tau(q)$,
    - Analytical $T(q)$
  - Choice of wavelet, of order $q$
  - Interpretation: what kind of spectrum did you estimate
    - Hölder exponent
    - Wavelet decay
To take away

• Cascades matured to versatile multifractal models

• There remains much to do.
Reading on this talk

- [link](www.stat.rice.edu/~riedi)
- This talk
- Intro for the “untouched mind”
  - Explicit computations on Binomial
- Monograph on “Multifractal processes”
  - Multifractal formalism (proofs)
  - Multifractal subordination (warping)
- Papers, links