# Scaling in Stochastic Processes

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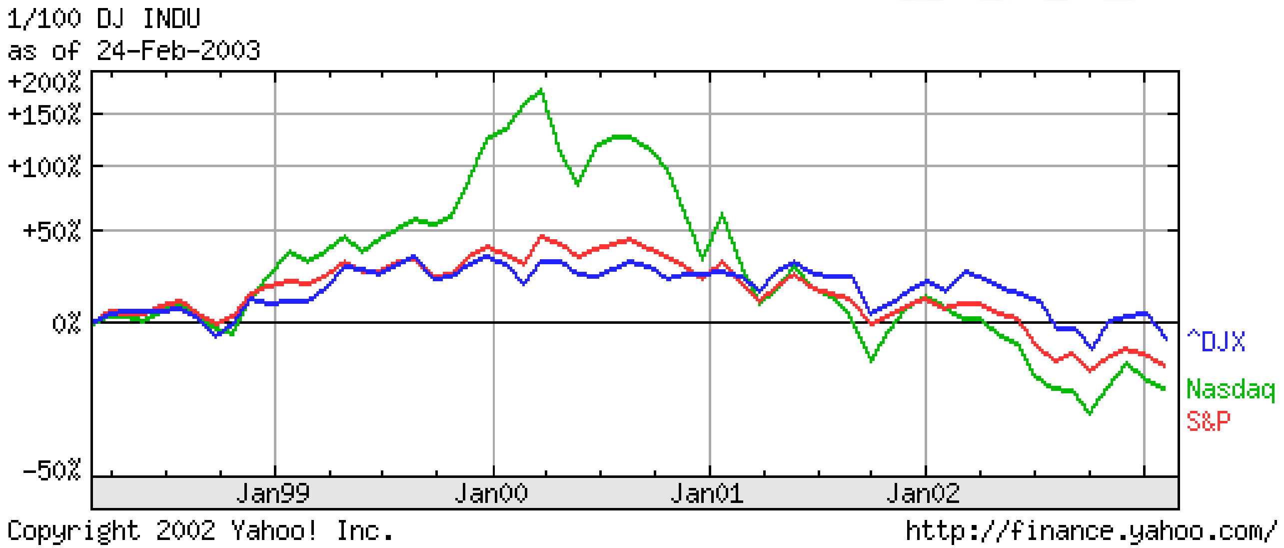
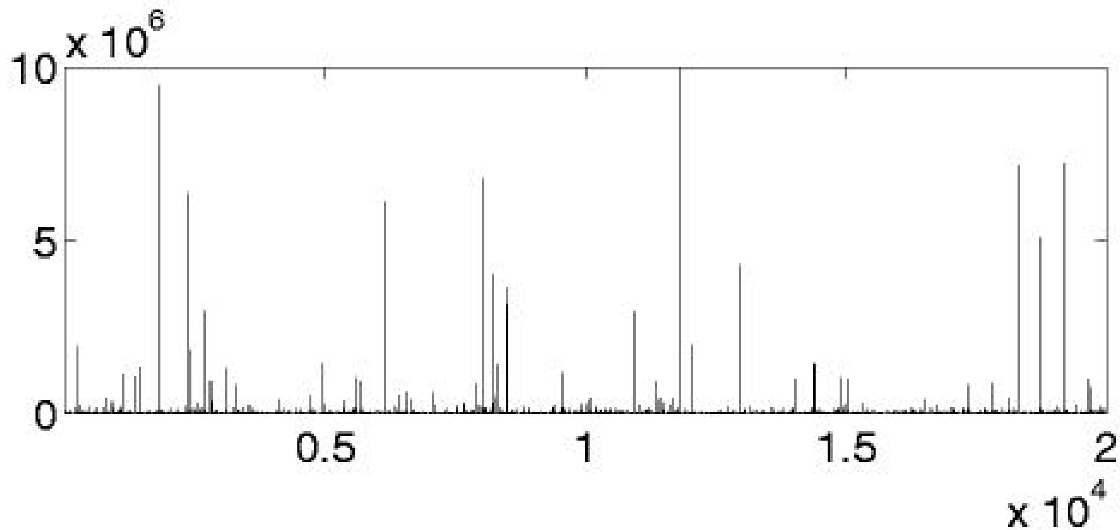
# Details Everywhere

DJX
Nasdaq
S&P

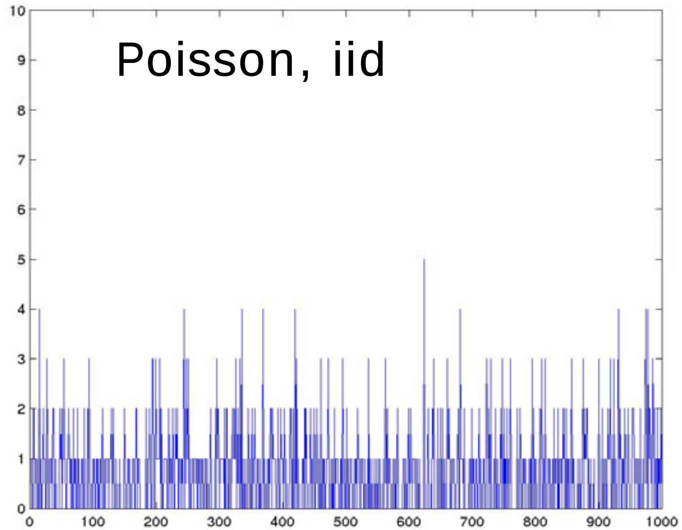
as of Feb 25 ‘03

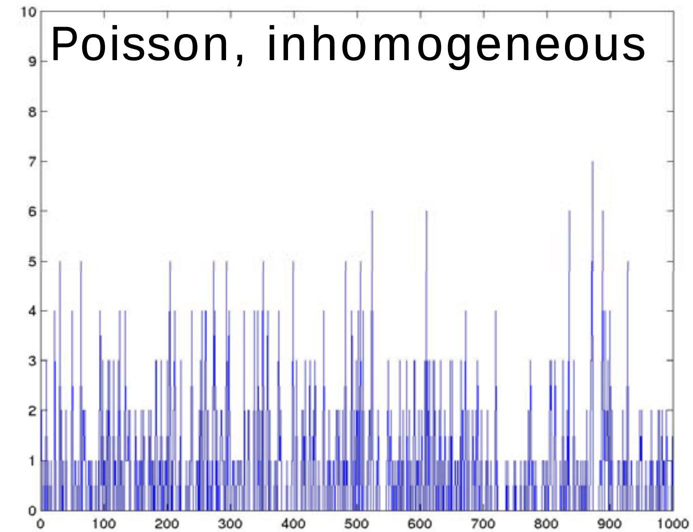



Request Load on E-commerce site



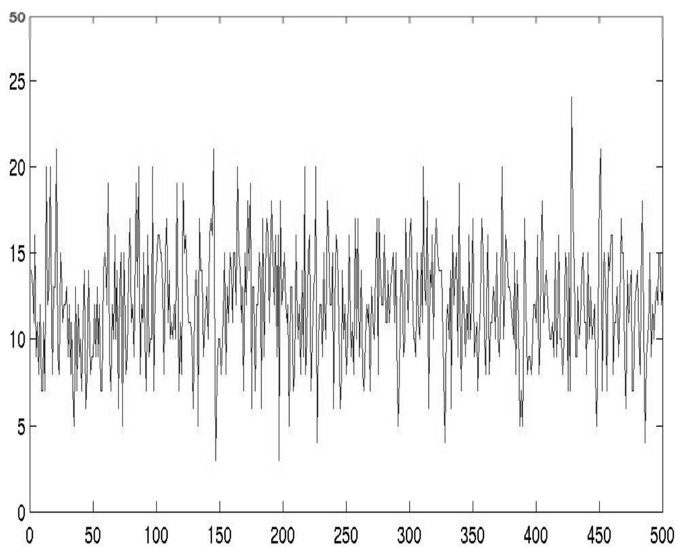
# Scaling and re-normalization



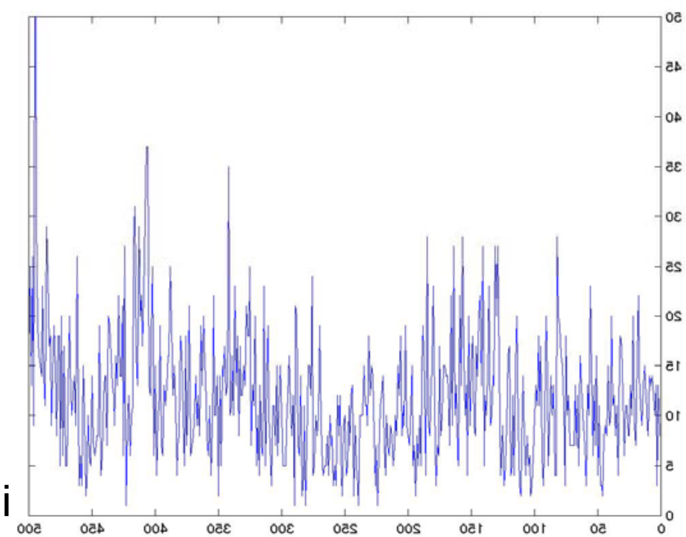




$X_k$

Sum up
16 neighbors



$Y_k = \Sigma_{i=0..15} X_{16k+i}$



# Scaling and re-normalization

- LLN:
  - Visual convergence of iid random variables to their mean: renormalize by "$1/n$"
- CLT
  - Visual convergence of iid random variables in distribution: renormalize by "$1/\sqrt{n}$"
  - For infinite variance, limit is stable: by "$n^{1/\alpha}$"
  - Also correlations may impose different exponent
  - Limiting distributions are invariant under aggregation
- Are there invariant processes?

# Statistical Self-similarity

- Self-similarity: canonical form
  - $B(at) =^{fdd} C(a)\, B(t)$ B: process, C: scale function

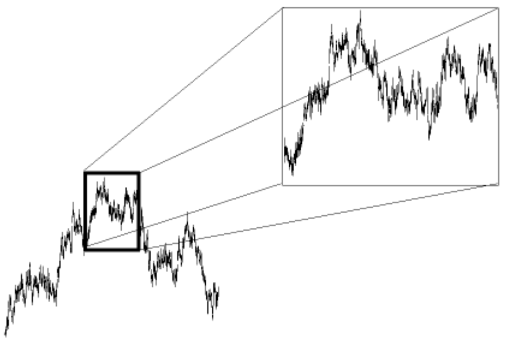
  - Iterate: $B(abt) =^{fdd} C(a)C(b)\, B(t)$
  - $C(a)C(b)=C(ab)$

  → $C(a) = a^H$ : Powerlaw is default

- H-self-similar:

  $$B(at) =^{fdd} a^H B(t)$$

  stationary increments



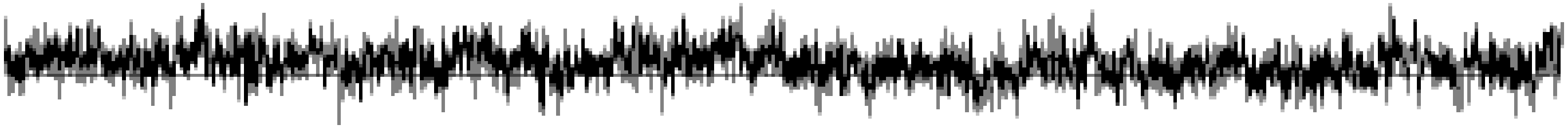
- Examples
  - Gaussian: unique, fractional Brownian motion
  - Stable: not unique in general, a=1/H: Levy motion

# Statistical Self-similarity

- How do self-similar processes occur?
  - $X_k$: stationary time series
  - $U(t) := X_1 + \ldots + X_{[t]}$
  - If $U(nt)/f(n) \to_{\text{f.d.d.}} Z(t)$
  - then necessarily $H = \lim_{n\to\infty} \log f(n) / \log(n)$ exists and $Z(t)$ is H-self-similar.
  - If $X_k$ are iid with finite variance, then $H=1/2$ and Z is Brownian motion
  - If $X_k$ are LRD, then $H>1/2$ and Z is fractional Brownian motion
- Prediction and estimation windows

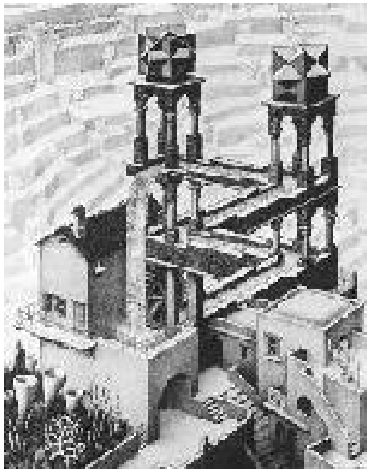
# Self-similar Processes

- What do they model?



  - Sustained excursions above/below the mean

- Different from (finite order) linear models
  - Auto-Regressive
  - ARMA
  - (G)ARCH
  - Exponential decay of correlations



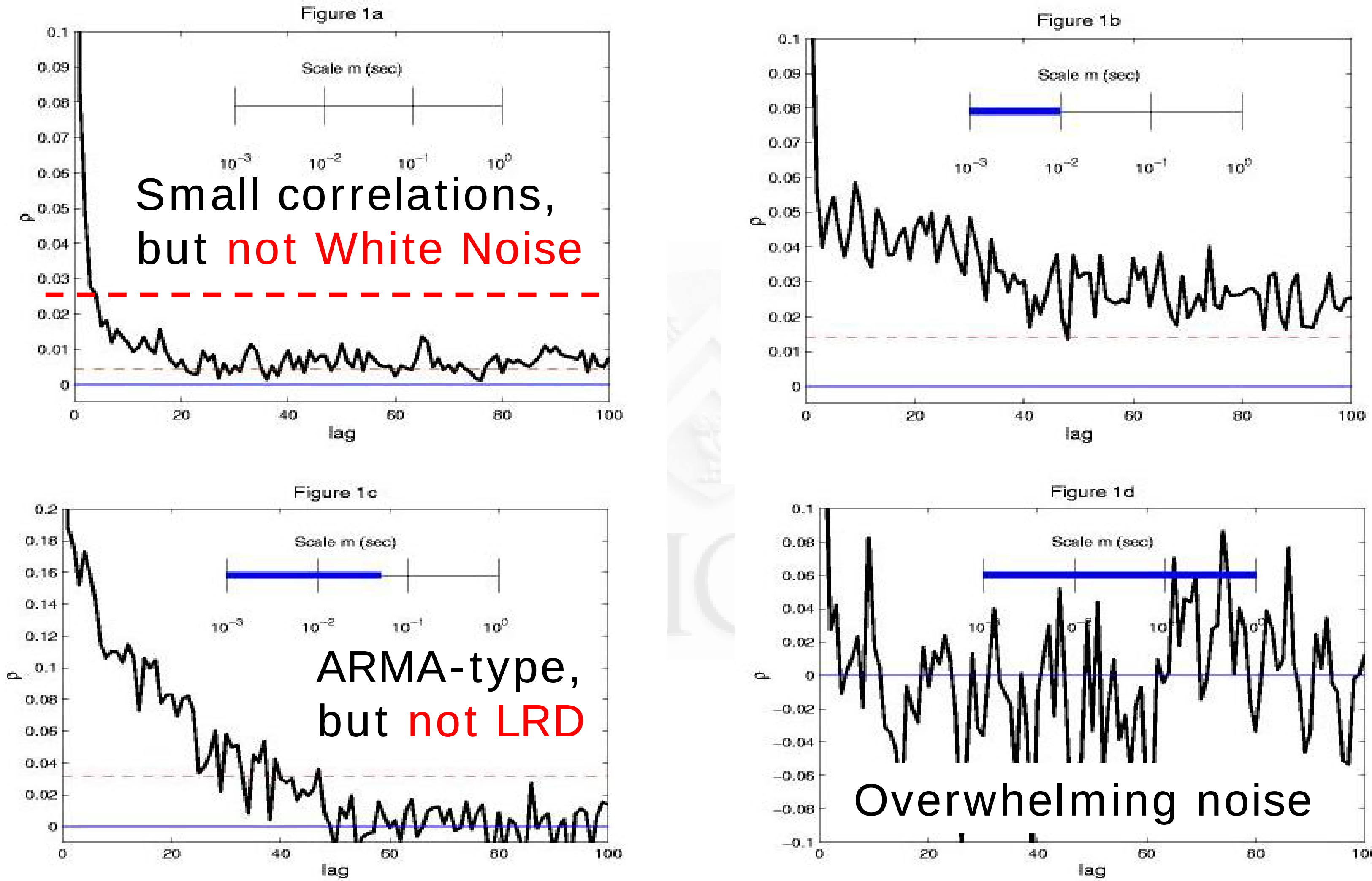
- Corresponds to infinite order AR models
  - FARIMA
  - FIGARCH

$$fBm(t) = \int_{-\infty}^{t} K(t,s)\, dW(s)$$

# Estimating LRD in network load

with Steve Marron, @Stat.UNC

Auto-correlation from 500 data points each, on increasing scales



# Auto-correlation: longer data set

10’000 data points, 20x longer than previous trace

Figure 2a

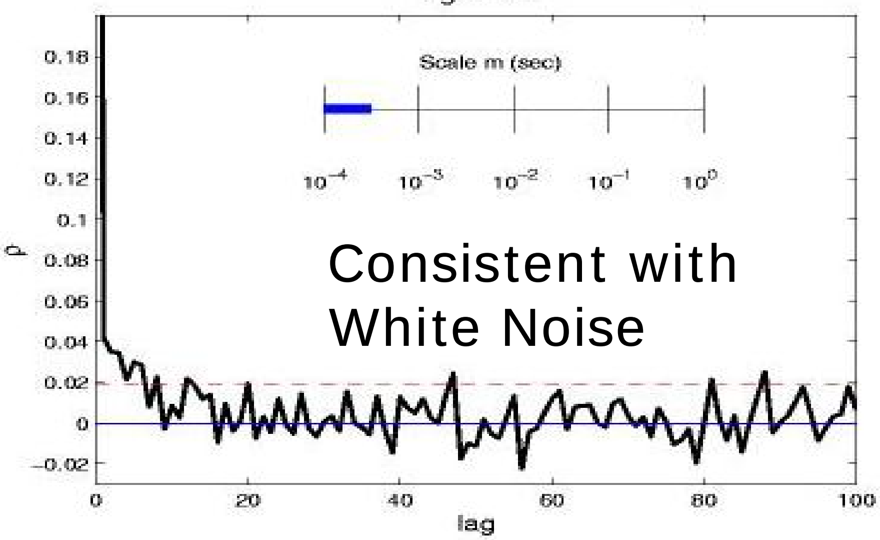



Figure 2b

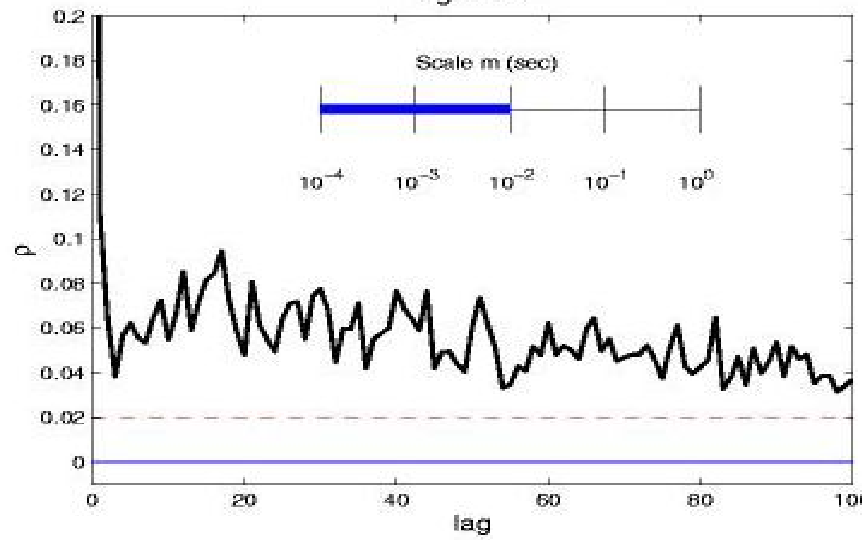



Figure 2c

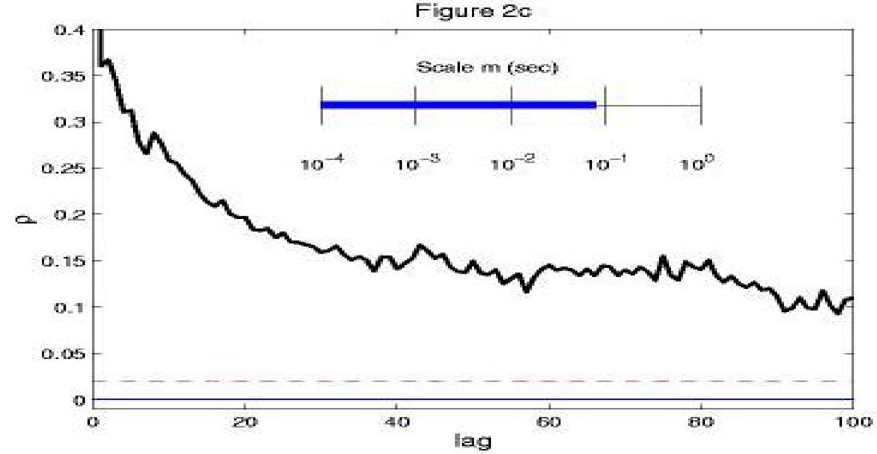
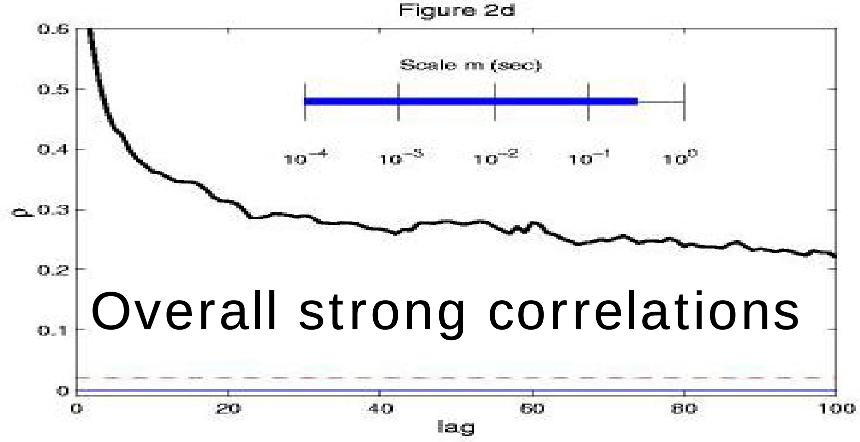



Figure 2d

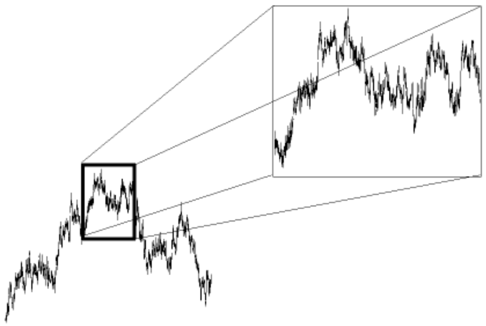


# Impact of Self-similar models

- Stock markets
  - Evertsz
  - Mandelbrot
  - Bacry, Muzy
- Networks
  - Willinger
  - Taqqu
- Turbulence
  - K62

# Limitation

- Scaling rigid
- Levy modulus of continuity:
  - $|B(t+s)-B(t)| \sim |s|^H$ (all t a.s.)
  - Elevated, yet constant volatility
- A formalism:
  - multifractal analysis



# Multifractal Bursts

# Multifractal formalism

- $X_j(k) = B((k+1)/2^j) - B(k/2^j)$:
  - increment process at resolution $1/2^j$

$U_j$ :Uniform on $\{1 \ldots 2^j\}$

- Chernoff bound :

$$\#\{k : |X_j(k)| > (2^j)^a\} = 2^j U_j[|X_j(k)|^q > 2^{jaq}] \quad (q>0)$$
$$\leq 2^j E_j[|X_j(k)|^q] / 2^{-jaq}$$
$$= 2^{jaq} \Sigma_k |X_j(k)|^q$$

- Large deviation principle $(j \to \infty)$

$$(1/j)\log E[\#\{k : |X_j(k)| > 2^{ja}\}] \to \inf_q (qa - T(q))$$

$T(q) = \lim -(1/j)\log E[\Sigma_k |X_j(k)|^q]$ : multiscale moments

# Mono-fractal Bursts

- $T(q) = \lim - (1/j) \log E[\Sigma_k \mid X_j(k) \mid^q]$
- Gaussian, H-self-similar:
  - $X_j(k) =_d X_j(1) = B(1/2^j) =_d 2^{-jH} B(1) \sim \mathcal{N}(0, \sigma^2 2^{-j2H})$
  - $E[\mid X_j(k) \mid^q] = 2^{-jqH} E[\mid B(1) \mid^q]$
  - $T(q) = qH - 1$ (number of terms in sum $\sim 2^j$)
- fBm is Mono-fractal!

$$(1/j) \log E[\#\{k : \mid X_j(k) \mid > 2^{ja}\}] \to \inf_q (\, qa - T(q) \,)$$

$$= 1 \quad \text{for } a = H$$

$$= -\infty \quad \text{otherwise}$$

# Multifractal Product of Pulses

together with
I. Norros and P. Mannersalo

# Beyond Self-similarity

- Recall: monofractal because self-similar→linear $T(q)$
- Self-similarity revisited:
  - $B(at) =^d C(a)\, B(t)$ B: process, C: scale function
  - $B(abt) =^d C(a)C(b)\, B(t)$
  - $C(a)C(b)=C(ab)$ → $C(a) = a^H$ : Powerlaw is default

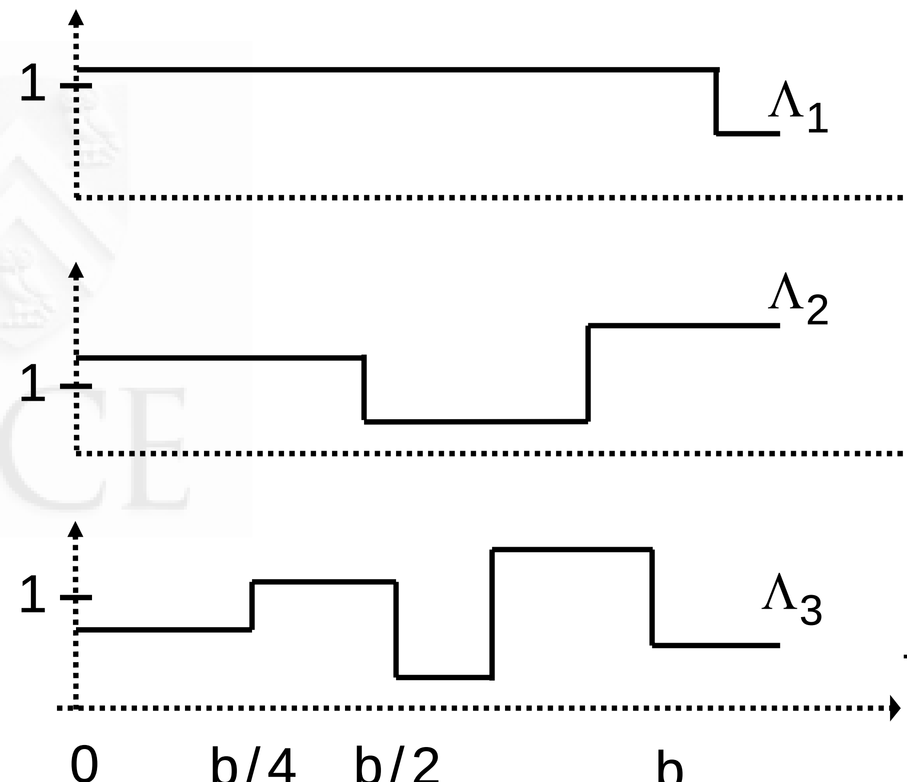
- More flexible rescaling "Ansatz":
  - $C=C(a,t)$ ?
  - C=independent r.v., one for every re-scaling:
  - $A(a...at)= A(a^n t) = C_n(a,t)...C_1(a,t)\, A(t)$
  - $E[|U_n(1)|^q]= E[|\, A(a^n)\, |^q] = c(q)\, E[|\, C(a)\, |^q]^{\,n}$
  - $T(q) = -1+\log_a E[|\, C\, |^q]$ : non-linear ! multifractal!

# Multifractal paradigm

- Multiplicative Processes:

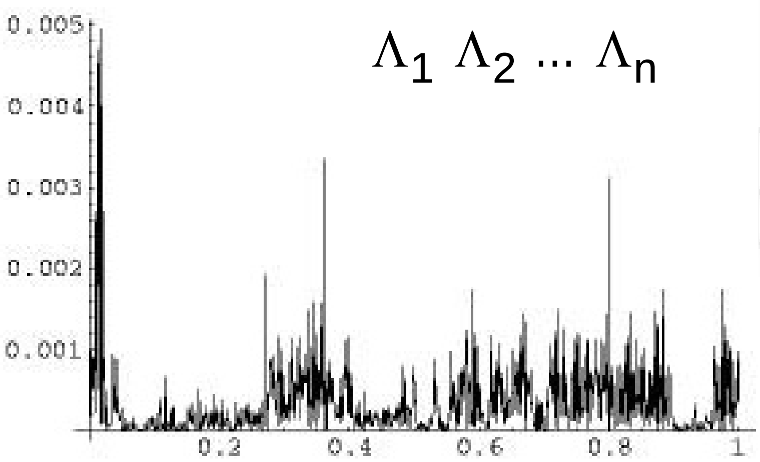
$$A(t) = \lim_{n\to\infty} \int_0^t \Lambda_1(s)\, \Lambda_2(s) \ldots \Lambda_n(s)\, ds$$

- Stationary Cascade
  - $\Lambda_n(s)$ is stationary
  - Conservation: $E\Lambda_n(t) = 1$
  - "self-similarity": $\Lambda_n(s) =_d \Lambda_1(sb^n)$



# Parameters and Scaling

- Parameter estimation
  - $\Lambda_i(s)$: i.i.d. values with Poisson arrivals $(\lambda_i)$:
  - $Z(s) = \log [\, \Lambda_1(s)\, \Lambda_2(s) \ldots \Lambda_n(s)\, ]$
  - $\mathrm{Cov}(Z(t)Z(t+s)) = \Sigma_{i=1..n} \exp(-\lambda_i s) \mathrm{Var}\, \Lambda_i(s)$
- Performance of predictors / simulations




$\Lambda_1\ \Lambda_2 \ldots \Lambda_n$

- Multifractal formalism

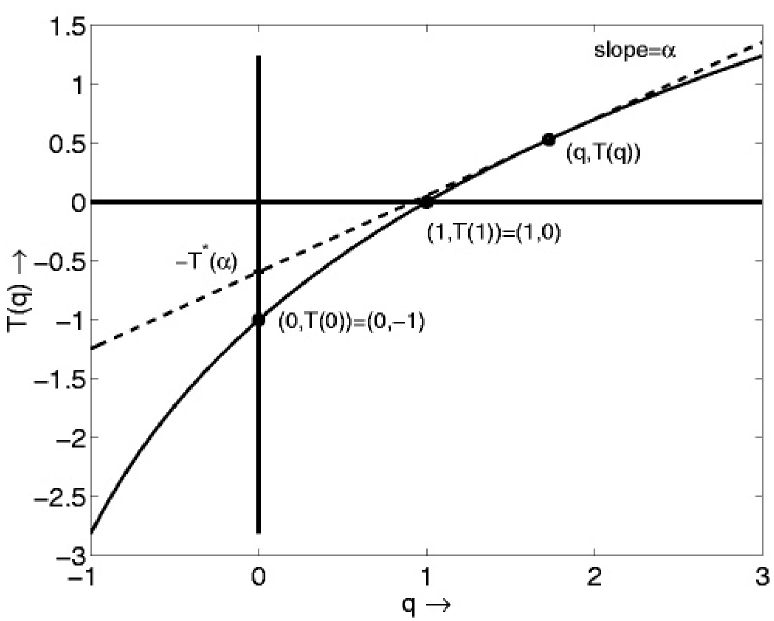
(with Norros and Mannersalo)

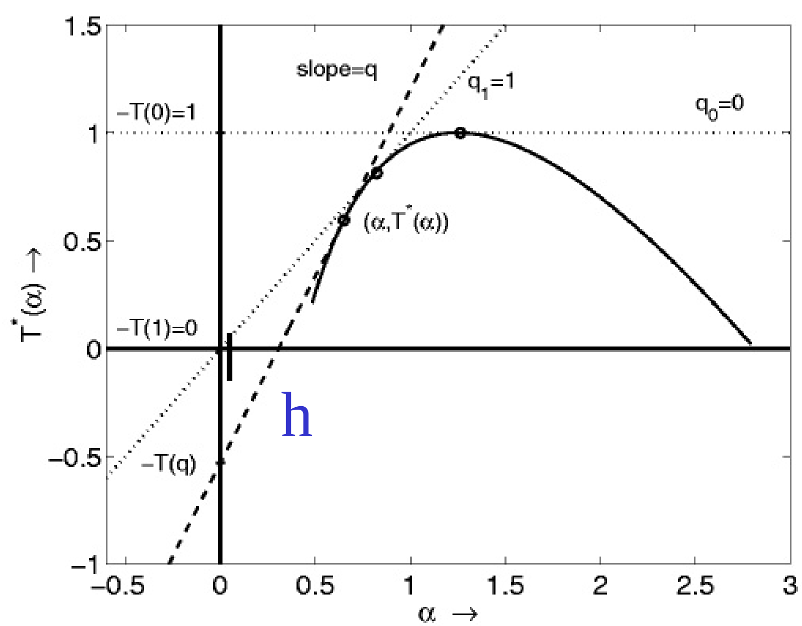
$$T(q) = q - 1 - \log_2 E[\Lambda^q]$$

# Kolmogorov criterion

- If $E[\ |\ A(s)-A(t)\ |^b\ ] < C\ |\ s-t\ |^{1+d}$ then almost all paths of A are of (global) Holder-continuity for all $h < d/b$ , i.e., for all $h < T(q)/q$ .
(smallest a of the spectrum T*(a) )

- Multifractal spectrum: local degree of continuity



# From
# Multiplicative Cascades
# to
# Infinitely Divisible Cascades

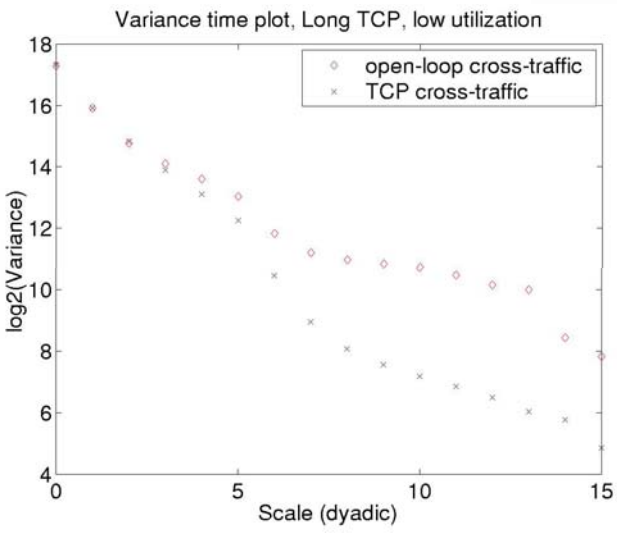
with

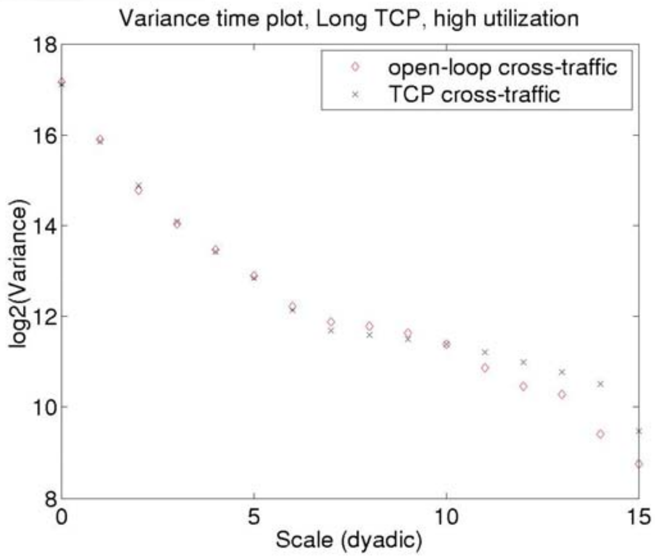
P. Chainais and P. Abry

# Beyond powerlaws

Real world data

- is stationary
- can deviate from powerlaws
- has no preference for dyadic scales



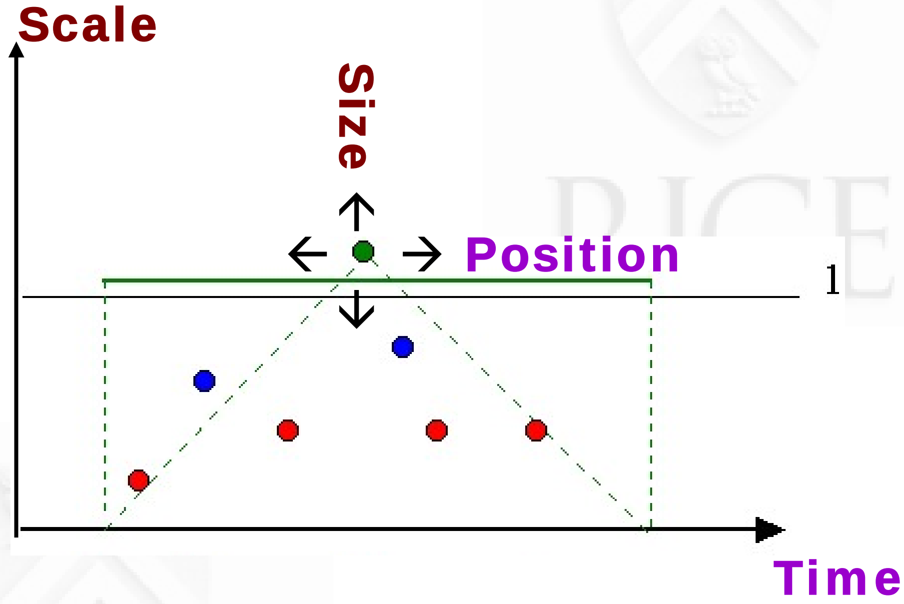
# Stationary Geometry

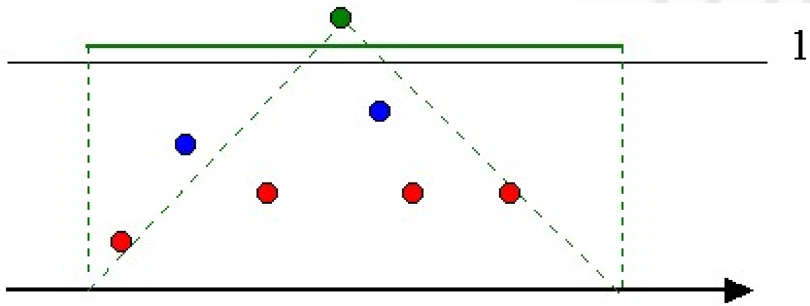
Scale-time plane (t,r)

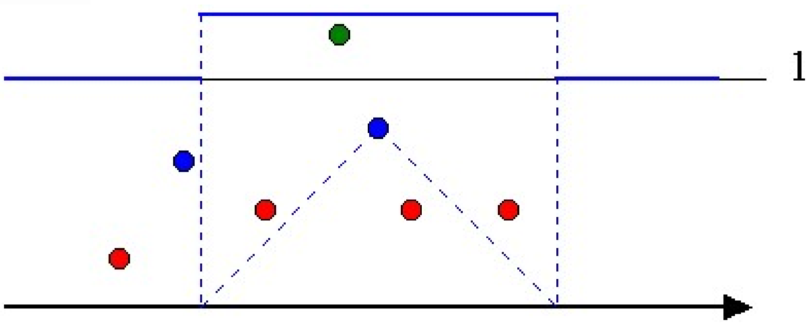
Pulses:

$P_i(t) = W_i$ if $|t-t_i|<r_i/2$

1 else



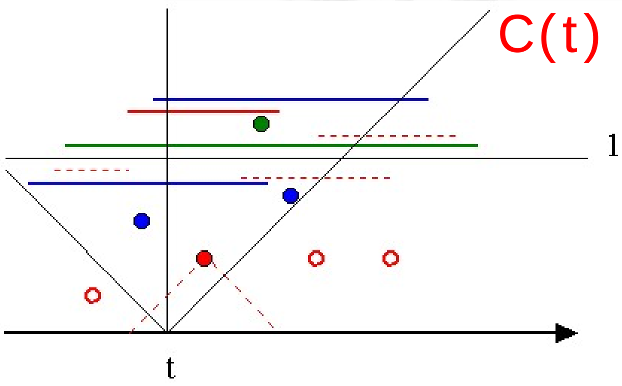



# Compound Poisson Cascade

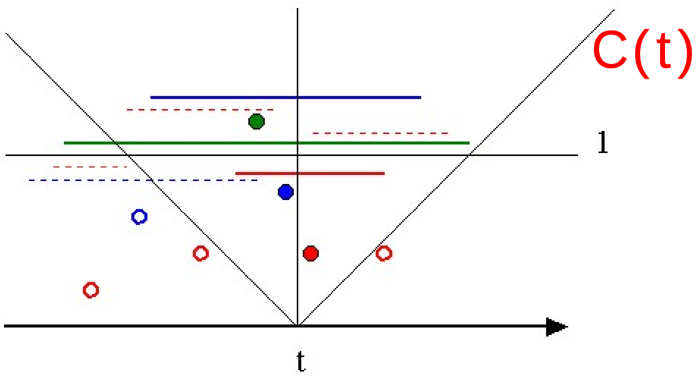
Poisson points with control measure m

Cone of influence at t

$C(t) = \{ (t_i, r_i) \ : \ |t - t_i| < r_i / 2 \ \}$

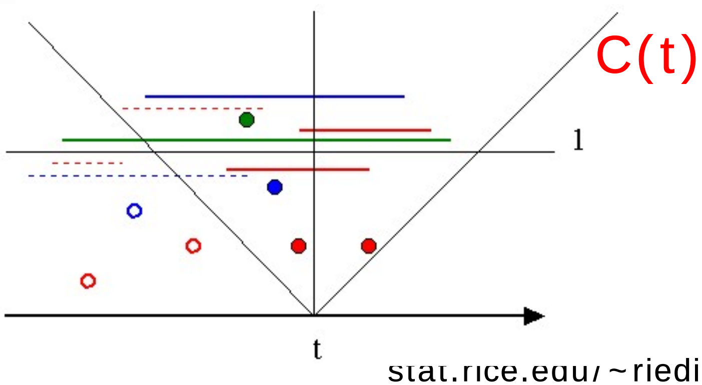



$Q(t) \ = \ \Pi \ \{W_i \ : \ |t - t_i| < r_i / 2 \ \}$




Multifractal Scaling

(with Abry & Chainais)




$T(q) \ = \ \exp[m(C(t,r))(1 - E[W^q])]$

→powerlaw only if $m(C(t,r)) \ = \ -\log(r)$

→More general Infinitely Divisible Laws

# Multifractal Subordination

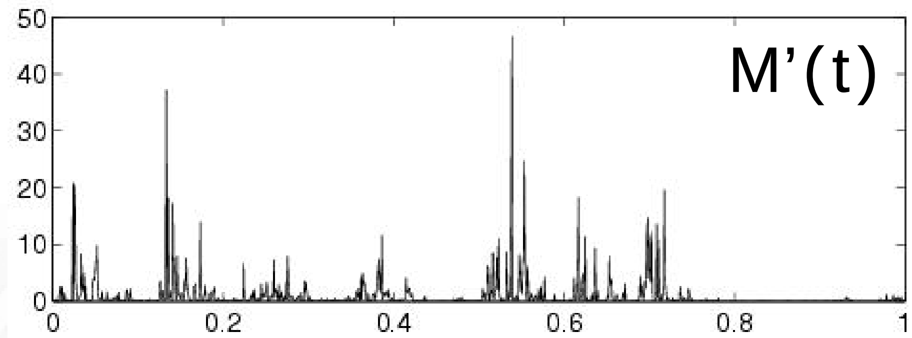
## Processes with multifractal oscillations

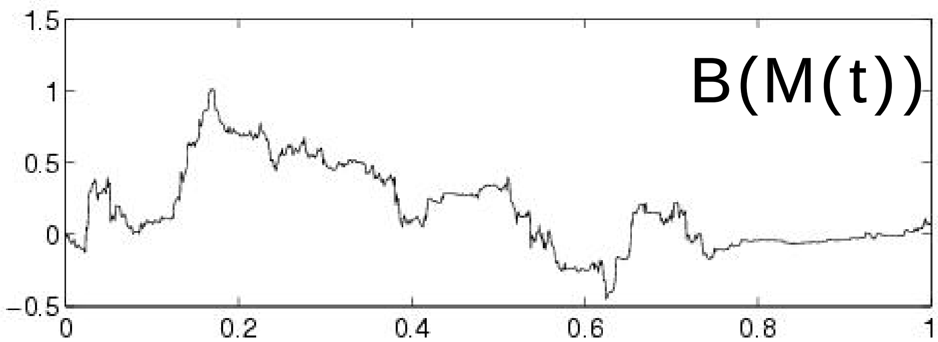
# Multifractal time warp

B(M(t)):
A versatile model

– M(t): Multifractal
Time change
Trading time

– B: Brownian motion
Gaussian fluctuations



# Auto-Correlation

- Conditioned on knowing M(t):
  - $E[\ B(M(t))\ B(M(s))\ |\ M\ ]$
    $= (\sigma^2/2)\ [\ M^{2H}(t) + M^{2H}(s) - M^{2H}(t-s)\ ]$
  - Non stationary Gaussian Process
  - Increments: $X(t)=B(M(t+1)) - B(M(t))$
  - $E[\ X(t)\ X(s)\ |\ M\ ] = (\sigma^2/2)\ x$

$( [M(t+1) - M(s)\ ]^{2H} - [\ M(t) - M(s)\ ]^{2H} - [M(t+1) - M(s+1)\ ]^{2H} + [\ M(t) - M(s+1)\ ]^{2H} )$

- Unconditioned: For $H=1/2$ and $E[M(t)]=t$
  - $E[\ B(M(t))\ B(M(s))\ ] = \sigma^2 \min(s,t)$
  - $E[\ X(t+k)\ X(t)\ ] = E[\ M(k+1) - 2M(k) + M(k-1)\ ]=0$
  - Uncorrelated increments, stationary, 2nd order
  - But not Gaussian
  - Dependence of higher order through M(t)

# Estimation: Wavelets decorrelate

- $W_{jk} = \int \psi_{jk}(t)\, B(M(t))\, dt$
  N: number of vanishing moments

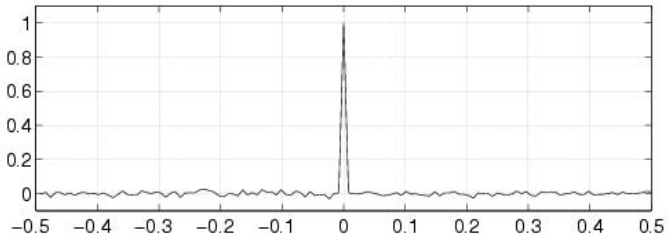
- $E[W_{jk} W_{jm}]$
  $= \int\int \Psi_{jk}(t)\, \Psi_{jm}(s)\, E[B(M(t))\, B(M(s))]\, dt\, ds$
  $= \int\int \Psi_{jk}(t)\, \Psi_{jm}(s)\, E[|M(t) - M(s)|^{2H}]\, dt\, ds$

  $\sim O( |k-m|^{T(2H) +1 - 2N} ) \quad ( |k-m| \rightarrow \infty)$

# Multifractal Estimation for B(M(t))

- Weak Correlations of Wavelet-Coefficients:
  (with Goncalves)

Haar

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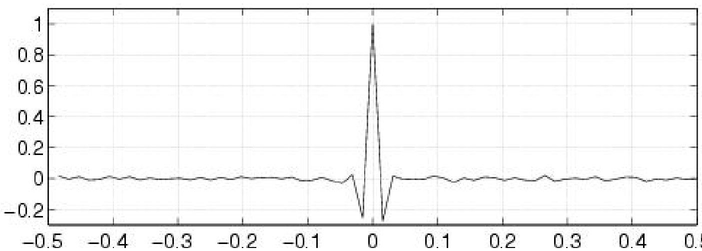


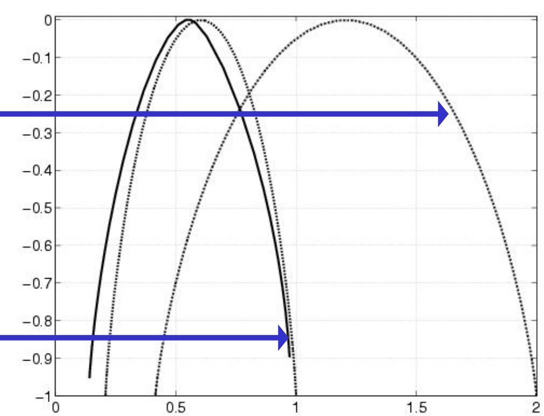

- Improved estimator due to weak correlations
- Multifraktal Spectrum

$M(t+s) - M(t) \sim s^{h(t)}$

$B(t+u) - B(t) \sim u^{H} \quad (\forall\, t)$

→

$B(M(t+s)) – B(M(t)) \sim s^{H*h(t)}$



# Volatility a time warp?

- IDC: causal rather than top-to-bottom
  - Prediction
  - Treat as a doubly stochastic estimation problem
  - Eg: multifractal statistics decouple
- Realism
  - Intense trading creating a time warp?
  - Scaling in finance
    - Evertsz, Mandelbrot,...
  - Multifractal random walk
    - Bacry, Muzy

# Testing for Diverging Moments

With Paulo Goncalves @INRIA.fr
All software freely available at
http://www.inrialpes.fr/is2/

# Scale-free Networks

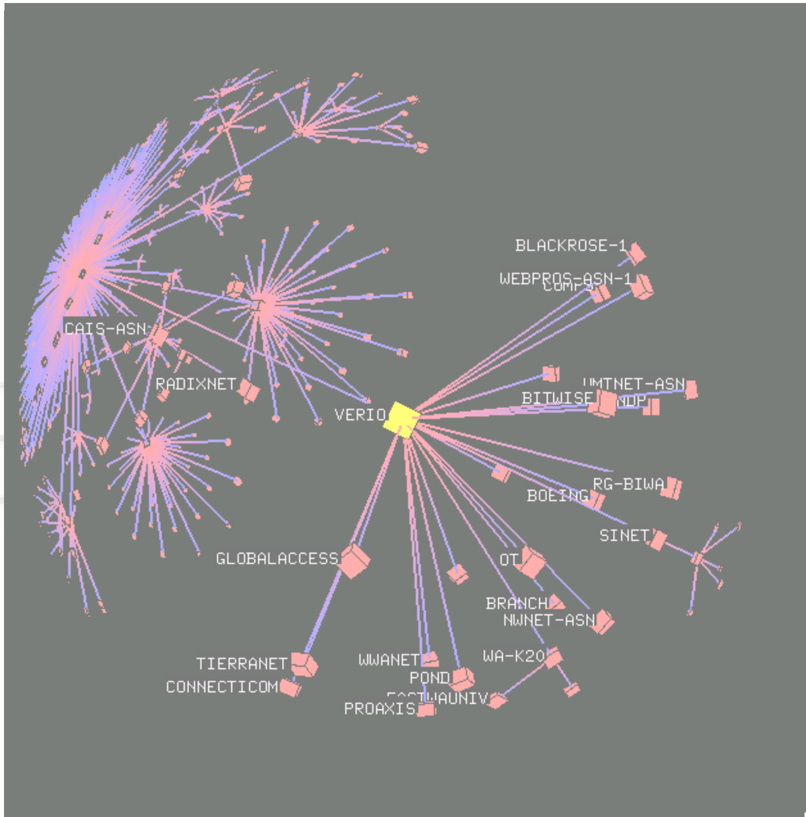
- Beginning of graph theory:
  - Königsberg (Euler, 1736),
  - Random graphs (Erdös-Renyi, 1959)
  - Virulence threshold
- Small world graphs
- Self-similarity
- Vulnerabilities

# Scale-free Networks

- Königsberg (Euler, 1736), Random graphs (Erdös-Renyi, 1959)
- Small world graphs:
  - 6 degrees of separation (Milgram, 1969)
  - Erdös Number
    - I am 2 handshakes and 3 papers away from Erdös
  - The strength of weak ties (Granovetter, 1973)
  - Clustering coefficient (Watts&Strogatz, 1998)
  - Synchronizing (fireflies)
- Self-similarity
- Vulnerabilities

# Scale-free Networks

- Königsberg (Euler, 1736), Random graphs (Erdös-Renyi, 1959)
- Small world: 6 degrees of separation, clustering, synchronizing
- Self-similarity
  – Pareto distribution of links per node
  – Critical state (phase transition)? No.
  – Preferential Attachment (rich get richer)
  → Hubs of all sizes
- Vulnerabilities



# Scale-free Networks

- Random graphs
- Small world: 6 degrees of separation, clustering, synchronizing
- Scale-free: preferential attachment, Hubs
- Vulnerabilities
  - Robust against failure, not against attack
  - No threshold for virulence: epidemics, viruses (AIDS, ILoveU)
  - Cancer growth
  - Cascading failures: power blackout 1996, Thai “baht” 1997
  - Corporate world
  - Out-sourcing, Hotmail.com, terror cells
  - Information passing through “Hubs”

    = increased trading time?

# Multiresolution approach

- Parsimonious description
- Efficient modeling and synthesis
- Physical relevance
- Non-parametric estimation
- Scale-free networks

*spin.rice.edu*

# References: Scaling processes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