I may have committed a number of errors in writing these solutions, but they should be fine for the most part. Use them at your own risk!

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**Problem 1**

a) Usual definitions

b) \( X_n = \begin{cases} n^2 w p_1/n^2 & \text{converges a.s. to zero by Borel-Cantelli and hence also in probability, but not in } L_1 \text{ since } E(X_n - 0) = E(X_n) = 1. \\ 0 & \text{with } X_n \text{ independent converges in } L_1 \text{ and thus in probability to zero, but not to zero by Borel-Cantelli.} \end{cases} \)

\( X_n = \begin{cases} 1 w p_1/n & \\ 0 w p_1 - 1/n \end{cases} \)

\( X = \begin{cases} 1 w p_1/n & Y = 1 - X. \text{ Obviously } Y \text{ and } X \text{ have the same distribution but } P(|X - Y| > 0) = 1, \text{ so we don’t have convergence in probability.} \end{cases} \)

**Problem 2**

a) (i) \( P(A\cap B) \geq 0 \) (ii) \( A_1 \ldots A_n \text{ disjoint,} \)
Problem 3

a) It means that (i) \( Y_n \) is \( \sigma(X_1...X_n) \) measurable and (ii) \( E(Y_{n+1}|\sigma(X_1...X_n)) = Y_n \).

b) 1) \( Y_n \) is a function of \( X_1...X_n \) and thus \( \sigma(X_1...X_n) \) measurable, and \( E(Y_{n+1}|\sigma(X_1...X_n)) = \{ \text{independent} \} = E(Y_{n+1}) = 0 \neq Y_n \), so it’s not a martingale.

2) \( Y_n \) is a function of \( X_1...X_n \) and thus \( \sigma(X_1...X_n) \) measurable, and \( E(Y_{n+1}|\sigma(X_1...X_n)) = \sum_{k=1}^{n} X_k + E(X_{n+1}) = \sum_{k=1}^{n} X_k = Y_n \), so it’s a martingale.

3) \( Y_n \) is NOT a function of \( X_1...X_n \) and thus NOT \( \sigma(X_1...X_n) \) measurable, so it’s not a martingale.

4) \( Y_n \) is a function of \( X_1...X_n \) and thus \( \sigma(X_1...X_n) \) measurable and \( E(Y_{n+1}|\sigma(X_1...X_n)) = \prod_{i=1}^{n} X_i E(X_{n+1}) = 0 \neq Y_n \), so not a martingale.

5) \( Y_n \) is a function of \( X_1...X_n \) and thus \( \sigma(X_1...X_n) \) measurable and \( E(Y_{n+1}|\sigma(X_1...X_n)) = \prod_{i=1}^{n} 2X_i E(2X_{n+1}) \), and \( E(2X_{n+1}) = \frac{1}{2} 2^{-1} + \frac{1}{2} 2 \neq 1 \), so it’s not a martingale.

Problem 4

By the Strong Law of Large Numbers, \( \sum X_i / n \) converges to 1/2 almost surely. For \( logG_n = \frac{1}{n} \sum logX_i \), since \( E(logX) = -1 \) (can be found integrating the pdf as usual) by SLLN again we have that \( logG_n \) converges almost surely to
−1. Hence by the Continuous Mapping Principle $G_n$ converges almost surely to $e^{-1}$.

Now since $E(1/X) = \infty$, by SLLN $\frac{1}{n} \sum_{i=1}^{n} \frac{1}{X_i} \to a.s. \infty$ and by CMP $\frac{1}{n} \sum_{i=1}^{n} \frac{1}{X_i} \to a.s. 0$. (note that SLLN also applies when the expectation is $\infty$.

Problem 5

1) The chain is irreducible since all states communicate and the period of all states is 1 since $p_{22} > 0$ and periodicity is a class property. To find $E(\tau_0)$ we’ll find the stationary distribution first. $\pi = \pi P$ gives $\pi_0 = 1/4$, $\pi_1 = 1/4$, $\pi_2 = 1/2$ and thus $E(\tau_0) = 1/\pi_0 = 4$.

2) a) This is a branching process, so $E(Z_n) = (2p)^n$.

b) We know that $\pi = 1$ iff $2p \leq 1$ so $p \leq 1/2$. Hence for $p > 1/2$ we got $\pi < 1$, and $\pi$ is the smallest solution to $\pi = G(\pi)$, where $G$ is the pgf of a binomial$(2, p)$. Solve for $s$ in $G(s) = (1 - p)^2 + 2p(1 - p)s + p^2s^2 = s$ to get $\pi = \left(\frac{1-p}{p}\right)^2$.

c) By independence, it’s $\pi^N$.

Problem 6

b) For $X$ continuous we know $F(X)$ is $Unif(0, 1)$ and hence $E(X) = 1/2$. For categorical consider $Y = F_{\chi}(X)$, and let $Z$ be any continuous random variable such that it’s cdf “matches” the cdf of $X$ in each of its jumps, i.e. $F_z(z) = P(Z \leq z) = P(X \leq z) = F_x(z)$ for all $z$ discontinuity point. Note that $F_z(z) \geq F_x(z)$, with strict equality in the discontinuity points (if this is confusing, plotting both cdfs may help to see it clearer). That gives us that $E(F_X(z)) \geq E(F_Z(z)) = 1/2$, the latter equality due to $Z$ being a continuous random variable.

c) I don’t know how to do it.
Problem 1

a) Usual definition

b) (i) Let $x \in (0, 1)$. $P(m_n \leq x) = 1 - P(m_n > x) = 1 - (P(X_1 > x))^n = 1 - (1 - x)^n \rightarrow 1$, i.e. $m_n \rightarrow^D 0$

(ii) $P(nm_n \leq x) = 1 - (P(X_1 > x/n))^n = 1 - (1 - x/n)^n \rightarrow 1 - e^{-x}$, which is the cdf of an exp(1).

c) Theorem. $\sum P(|X_n - X| > \epsilon) < \infty \Rightarrow X_n \rightarrow_{a.s.} X$

Since $\sum P(|m_n - 0| > \epsilon) = \sum P(m_n > \epsilon) = \sum (1 - \epsilon)^n < \infty$, we have $m_n \rightarrow_{a.s.} 0$. In general $nm_n$ doesn’t converge almost surely to $X$, although by Skorohod’s theorem we know that $\exists Y_n =^D nm_n$, $Y =^D X$ such that $Y_n \rightarrow Y$.

Problem 2

a) Delta method.

b) Let’s find first the asymptotic distribution of $logG_n = \frac{1}{n} \sum logX_i$. It can be seen that $E(logX_i) = -\frac{1}{2}$ and $V(logX_i) = \frac{1}{2}$, so by the Central Limit Theorem $\sqrt{n} \left( logG_n + \frac{1}{2} \right) \rightarrow^D N(0, 1/4)$, and since $g(x) = e^x \Rightarrow g'(x) = e^x$ the delta method gives that $\sqrt{n} \left( G_n - e^{-1/2} \right) \rightarrow^D N(0, e/4)$

Now, for $H_n$ let’s first find the asymptotic distribution of $H_n^{-1} = \frac{1}{n} \sum \frac{1}{X_k}$. Since $E(1/X) = 2$, $V(1/X) = \infty$ the CLT doesn’t apply to $H_n^{-1}$.

Problem 3

a) BCI and BCI I

b) (i) $X_n = \begin{cases} n^2 & wp1/n^2 \\ 0 & wp1 - 1/n^2 \end{cases}$ independent.
(ii) \( X_n = \left\{ n - 1 \begin{array}{c} \frac{wp1/n}{0} \\ \frac{wp1 - 1/n}{0} \end{array} \right\} \) doesn’t converge in \( L_1 \) nor a.s., but it does in probability.

(iii) \( X_n = \left\{ \frac{n}{-n(n^2 - 1)} \begin{array}{c} \frac{wp1 - 1/n^2}{wp1/n^2} \end{array} \right\} \), by BCI \( X_n \to_{a.s.} -\infty \), but \( E(X_n) = 0 \).

(iv) \( X_n = \left\{ \frac{n}{-n^4} \begin{array}{c} \frac{wp1 - 1/n^2}{wp1/n^2} \end{array} \right\} \). By BCI \( X_n \to_{a.s.} 0 \) but \( E(X_n) \to -\infty \).

**Problem 4**

1) (a) \( \{0\} \) is transient and \( \{1\}, \{2\} \) are recurrent.

(b) We can find the stationary distribution of the Markov Chain defined by \( \{1, 2\} \),

\[
P = \left( \begin{array}{cc} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{array} \right)
\]

and \( \pi = \pi P \) gives \( \pi_1 = 1/3 \) and \( \pi_2 = 2/3 \). Hence \( E(N|X_0 = 2) = 3/2 \).

(c) Let \( Y \) be the number of transitions until we leave 0. \( P(Y = y) = \frac{2}{3} \left( \frac{1}{3} \right)^{y-1} \) for \( y = 1, 2, 3... \) i.e. \( Y \sim Geom(2/3) \). Hence \( E(Y) = 3/2 \).

2) Not necessary

**Problem 5**

a) State MCT and DCT

b) (i) \( \int_1^n f_{[n,n+1]}(x)dx = \int_n^{n+1} dx = 1 \forall n \)

(ii) Since \( \frac{\sin(nx)}{e^{nx^2}} \leq \frac{1}{e^{nx^2}} \) which is integrable, DCT gives \( \int_1^n \frac{\sin(nx)}{e^{nx^2}} \frac{1}{x^2}dx \to \int_1^\infty \frac{1}{x^2} \lim_{n \to \infty} \frac{\sin(nx)}{e^{nx^2}} \frac{1}{x^2}dx = 0 \)

(iii) \( \int_1^n x/n dx = \frac{1}{n} \int_1^n x dx = \infty \forall n \)

(iv) For \( n \) odd, \( -\int_1^n x/n^2 dx = -\frac{1}{n^3} \infty = -\infty \forall n \), and for \( n \) even \( \int_1^n x/n^2 dx = \frac{1}{n^3} \infty = \infty \forall n \), so the limit doesn’t exist.
The function inside the integral is positive and decreasing with \( n \), with its limit being 0 so for \( n \geq 2 \)

\[
\left| \frac{(1+nx^2)}{(1+x^2)^n} \right| < \frac{1+2x^2}{(1+x^2)^2}
\]

which is integrable since the degree of the denominator is 4 and for the numerator it's 2. Hence DCT applies to give

\[
\int_1^{\infty} \frac{(1+nx^2)}{(1+x^2)^n} \, dx \to \int_1^{\infty} \lim \frac{(1+nx^2)}{(1+x^2)^n} \, dx = 0.
\]

**Problem 6**

a) \( Y_n \) is a function of \( X_1...X_n \) \( \Rightarrow \) \( \sigma(X_1...X_n) \) measurable. Some algebra shows that

\[
E(S^4_{n+1}|\sigma(X_1...X_n)) = S^4_n + 6S^2_n + 1, \quad E(S^2_{n+1}|\sigma(X_1...X_n)) = S^2_n + 1.
\]

Thus

\[
E(Y_{n+1}|\sigma(X_1...X_n)) = E(S^4_{n+1} - 6(n+1)S^2_{n+1} + 3(n+1)^2 + 2(n+1)|\sigma(X_1...X_n)) = ...
\]

\( S^4_n - 6nS^2_n + 3n^2 + 2n = Y_n. \)

Hence \( \{Y_n\} \) is a martingale.

b) The optional stopping theorem gives that \( Y_o, Y_\nu \) is a martingale so in particular

\[
E(Y_\nu) = E(Y_o) = 0. \quad \text{Since } E(Y_\nu) = E(S^4_\nu - 6\nu S^2_\nu + 3\nu^2 + 2\nu) = ...
\]

\( -5a^4 + 2a^2 + 3E(\nu^2), \) we get that

\[
E(\nu^2) = \frac{5a^4 - 2a^2}{3} \quad \text{and } V(\nu) = \frac{2}{3}a^2(a^2 - 1).
\]