Probability A exam solutions

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I may have committed a number of errors in writing these solutions, but they should be fine for the most part. Use them at your own risk!

Probability January 2003

Problem 1

a) Usual definitions

b) $X_n = \begin{cases} n^2 wp 1/n^2 \\ 0 wp 1-1/n^2 \end{cases}$ converges a.s. to zero by Borel-Cantelli and hence also in probability, but not in L_1 since $E(X_n - 0) = E(X_n) = 1$.

$X_n = \begin{cases} 1 wp 1/n \\ 0 wp 1-1/n \end{cases}$ with X_n independent converges in $L_1$and thus in probability to zero, but not to zero by Borel-Cantelli.

$X = \begin{cases} 1 wp 1/n \\ 0 wp 1-1/n \end{cases}$, $Y = 1 - X$. Obviously Y and X have the same distribution but $P(|X-Y| > 0) = 1$, so we don't have convergence in probability.

Problem 2

a) (i) $P(A \bigcap B) \geq 0)$ (ii) $A_1 ... A_n$ disjoint,

$Q\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \bigcap B\right) = P\left(\bigcup_{i=1}^{\infty} (A_i \bigcap B)\right) = \{disjoint\} = \sum_{i=1}^{\infty} P(A_i \bigcap B) = \sum_{i=1}^{\infty} Q(A_i)$

For Q to be a probability measure, we need $Q(\Omega) = P(\Omega \bigcap B) = P(B) = 1$.

b) Start with indicators

(i) $X = I_A$. Then $E(X;B) = \int_B I_A dP = P(A \bigcap B) = Q(A) = \int_\Omega X dQ$

(ii) $X = \sum_{i=1}^n a_k I_{A_k}$. Then $\int_\Omega X dQ = \int_\Omega \sum_{i=1}^n a_k I_{A_k} dQ = \sum_{i=1}^n a_k \int_\Omega I_{A_k} dQ = \sum a_k P(A_k B) = \int_B \sum_{i=1}^n a_k I_{A_k} dP = E(X;B)$.

(iii) $X \geq 0$. Let $0 \leq X_n \uparrow X$ with $X_n$simple and $\int_\Omega X dQ = ... = \int_B X dP$

(iv) General X. Proceed as usual.

Problem 3

a) It means that (i) Y_n is $\sigma(X_1...X_n)$ measurable and (ii) $E(Y_{n+1}|\sigma(X_1...X_n)) = Y_n$.

b) 1) $Y_n$is a function of $X_1...X_n$and thus $\sigma(X_1...X_n)$ measurable, and $E(Y_{n+1}|\sigma(X_1...X_n)) = \{independent\} = E(Y_{n+1}) = 0 \neq Y_n$, so it's not a martingale.

2) $Y_n$is a function of $X_1...X_n$and thus $\sigma(X_1...X_n)$ measurable, and $E(Y_{n+1}|\sigma(X_1...X_n)) = \sum_{k=1}^n X_k + E(X_{n+1}) = \sum_{k=1}^n X_k = Y_n$, so it's a martingale.

3) $Y_n$is NOT a function of $X_1...X_n$and thus NOT $\sigma(X_1...X_n)$ measurable, so it's not a martingale.

4) $Y_n$is a function of $X_1...X_n$and thus $\sigma(X_1...X_n)$ measurable and $E(Y_{n+1}|\sigma(X_1...X_n)) = \prod_{i=1}^n X_i E(X_{n+1}) = 0 \neq Y_n$, so not a martingale.

5) $Y_n$is a function of $X_1...X_n$and thus $\sigma(X_1...X_n)$ measurable and $E(Y_{n+1}|\sigma(X_1...X_n)) = \prod_{i=1}^n 2^{X_i} E(2^{X_{n+1}})$, and $E(2^{X_{n+1}}) = \frac{1}{2}2^{-1} + \frac{1}{2}2 \neq 1$, so it's not a martingale.

Problem 4

By the Strong Law of Large Numbers, $\sum X_i/n$ converges to $1/2$ almost surely. For $logG_n = \frac{1}{n}\sum logX_i$, since $E(logX) = -1$ (can be found integrating the pdf as usual) by SLLN again we have that $logG_n$ converges almost surely to

-1. Hence by the Continuous Mapping Principle G_n converges almost surely to e^{-1}.

Now since $E(1/X) = \infty$, by SLLN $\frac{1}{n}\sum \frac{1}{X_i} \rightarrow_{a.s.} \infty$ and by CMP $\frac{n}{\sum 1/X_i} \rightarrow_{a.s.} 0$. (note that SLLN also applies when the expectation is ∞.

Problem 5

1) The chain is irreducible since all states communicate and the period of all states is 1 since $p_{22} > 0$ and periodicity is a class property. To find $E(\tau_0)$ we'll find the stationary distribution first. $\pi = \pi P$ gives $\pi_0 = 1/4$, $\pi_1 = 1/4$, $\pi_2 = 1/2$ and thus $E(\tau_0) = 1/\pi_0 = 4$.

2) a)This is a branching process, so $E(Z_n) = (2p)^n$.

b) We know that $\pi = 1$ iff $2p \leq 1$ so $p \leq 1/2$. Hence for $p > 1/2$ we got $\pi < 1$, and π is the smallest solution to $\pi = G(\pi)$, where G is the pgf of a binomial(2,p). Solve for s in$G(s) = (1-p)^2 + 2p(1-p)s + p^2s^2 = s$ to get $\pi = \left(\frac{1-p}{p}\right)^2$.

c) By independence, it's π^N.

Problem 6

b) For X continuous we know $F(X)$ is $Unif(0,1)$ and hence $E(X) = 1/2$. For X categorical consider $Y = F_x(X)$, and let Z be any continuous random variable such that it's cdf "matches" the cdf of X in each of its jumps, i.e. $F_z(z) = P(Z \leq z) = P(X \leq z) = F_x(z)$ for all z discontinuity point. Note that $F_z(z) \geq F_x(z)$, with strict equality in the discontinuity points (if this is confusing, plotting both cdfs may help to see it clearer). That gives us that $E(F_X(z)) \geq E(F_Z(z)) = 1/2$, the latter equality due to Z being a continuous random variable.

c) I don't know how to do it.

Probability January 2004

Problem 1

a) Usual definition

b) (i) Let $x \in (0,1)$. $P(m_n \leq x) = 1 - P(m_n > x) = 1 - (P(X_1 > x))^n = 1 - (1-x)^n \to 1$, i.e. $m_n \to^D 0$

(ii) $P(nm_n \leq x) = 1 - (P(X_1 > x/n))^n = 1 - (1 - x/n)^n \to 1 - e^{-x}$, which is the cdf of an $exp(1)$.

c) Theorem. $\sum P(|X_n - X| > \epsilon) < \infty \Rightarrow X_n \to_{a.s.} X$

Since $\sum P(|m_n - 0| > \epsilon) = \sum P(m_n > \epsilon) = \sum (1-\epsilon)^n < \infty$, we have $m_n \to_{a.s.} 0$. In general nm_n doesn't converge almost surely to X, although by Skorohod's theorem we know that $\exists Y_n =^D nm_n$, $Y =^D X$ such that $Y_n \to Y$.

Problem 2

a) Delta method.

b) Let's find first the asymptotic distribution of $logG_n = \frac{1}{n} \sum logX_i$. It can be seen that $E(logX_i) = -\frac{1}{2}$ and $V(logX_i) = \frac{1}{4}$, so by the Central Limit Theorem $\sqrt{n}\left(logG_n + \frac{1}{2}\right) \to^D N(0, 1/4)$, and since $g(x) = e^x \Rightarrow g'(x) = e^x$the delta method gives that $\sqrt{n}\left(G_n - e^{-1/2}\right) \to^D N(0, e/4)$

Now, for $H_n$let's first find the asymptotic distribution of $H_n^{-1} = \frac{1}{n} \sum \frac{1}{X_k}$. Since $E(1/X) = 2$, $V(1/X) = \infty$ the CLT doesn't apply to H_n^{-1}.

Problem 3

a) BCI and BCII

b) (i) $X_n = \left\{ \begin{array}{cc} n^2 & wp1/n^2 \\ 0 & wp1 - 1/n^2 \end{array} \right\}$ independent.

(ii) $X_n = \left\{ \begin{array}{cc} n-1 & wp1/n \\ 0 & wp1-1/n \end{array} \right\}$ doesn't converge in L_1 nor a.s., but it does in probability.

(iii) $X_n = \left\{ \begin{array}{cc} n & wp1-1/n^2 \\ -n(n^2-1) & wp1/n^2 \end{array} \right\}$, by BCI $X_n \to_{a.s.} -\infty$, but $E(X_n) = 0$.

(iv) $X_n = \left\{ \begin{array}{cc} n & wp1-1/n^2 \\ -n^4 & wp1/n^2 \end{array} \right\}$. By BCI $X_n \to_{a.s.} 0$ but $E(X_n) \to -\infty$.

Problem 4

1) (a) $\{0\}$ is transient and $\{1\},\{2\}$ are recurrent.

(b) We can find the stationary distribution of the Markov Chain defined by $\{1,2\}$,

$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $\pi = \pi P$ gives $\pi_1 = 1/3$ and $\pi_2 = 2/3$. Hence $E(N|X_0 = 2) = 3/2$.

(c) Let Y be the number of transitions until we leave 0. $P(Y=y) = \frac{2}{3}\left(\frac{1}{3}\right)^{y-1}$ for $y = 1,2,3...$ i.e. $Y \sim Geom(2/3)$. Hence $E(Y) = 3/2$.

2) Not necessary

Problem 5

a) State MCT and DCT

b) (i) $\int_1^\infty I_{[n,n+1]}(x)dx = \int_n^{n+1} dx = 1 \ \forall n$

(ii) Since $\left|\frac{sin(nx)}{e^{nx}x^2}\right| \leq \frac{1}{e^x x^2}$ whic is integrable, DCT gives $\int_1^\infty \frac{sin(nx)}{e^{nx}}\frac{1}{x^2}dx \to \int_1^\infty \frac{1}{x^2} lim \frac{sin(nx)}{e^{nx}}dx = 0$

(iii) $\int_1^\infty x/n dx = \frac{1}{n}\int_1^\infty x dx = \infty \ \forall n$

(iv) For n odd, $-\int_1^\infty x/n^2 dx = -\frac{1}{n^2}\infty = -\infty \ \forall n$, and for n even $\int_1^\infty x/n^2 dx = \frac{1}{n^2}\infty = \infty \ \forall n$, so the limit doesn't exist.

(v) The function inside the integral is positive and decreasing with n, with its limit being 0 so for $n \geq 2$

$\left|\frac{(1+nx^2)}{(1+x^2)^n}\right| < \frac{1+2x^2}{(1+x^2)^2}$ which is integrable since the degree of the denominator is 4 and for the numerator it's 2. Hence DCT applies to give $\int_1^\infty \frac{(1+nx^2)}{(1+x^2)^n}dx \to \int_1^\infty lim\frac{(1+nx^2)}{(1+x^2)^n}dx = 0$.

Problem 6

a) Y_n is a function of $X_1...X_n \Rightarrow \sigma(X_1...X_n)$ measurable. Some algebra shows that $E(S_{n+1}^4|\sigma(X_1...X_n)) = S_n^4 + 6S_n^2 + 1$, $E(S_{n+1}^2|\sigma(X_1...X_n)) = S_n^2 + 1$. Thus $E(Y_{n+1}|\sigma(X_1...X_n)) = E(S_{n+1}^4 - 6(n+1)S_{n+1}^2 + 3(n+1)^2 + 2(n+1)|\sigma(X_1...X_n)) = ... = S_n^4 - 6nS_n^2 + 3n^2 + 2n = Y_n$.

Hence $\{Y_n\}$is a martingale.

b) The optional stopping theorem gives that Y_o, Y_ν is a martingale so in particular $E(Y_\nu) = E(Y_o) = 0$. Since $E(Y_\nu) = E(S_\nu^4 - 6\nu S_\nu^2 + 3\nu^2 + 2\nu) = ... = -5a^4 + 2a^2 + 3E(\nu^2)$, we get that

$E(\nu^2) = \frac{5a^4-2a^2}{3}$ and $V(\nu) = \frac{2}{3}a^2(a^2-1)$.