Stat 310 SECOND EXAM March 23, 2001 Dr. Scott

Instructions:
1) Take home; time limit 2 hours, 30 minutes; record time/date you took the exam; Due in class Tuesday, March 27. SIGN PLEDGE.
2) Materials allowed: class book/notes/homework, calculator, and math books.
3) Show all your work. Use a separate sheet of paper for each question, and STAPLE together in proper order. Work quickly.
4) There are 8 problems totaling 125 points. All problems will be graded. Any points over 100 will count 1/5 (maximum 105, rounded up).
5) If you find an error or typo, please make a reasonable correction and proceed.

1. [*Binomial Random Variable*] A poll of 900 voters may be modeled by a Binomial, $B(n,p)$, distribution, with $n = 900$ and $p$ unknown. A congressional aide believes $p = 0.65$ in support of the repeal of the marriage tax. Her pollster reports that of 900 voters asked, 550 supported the marriage tax repeal and 350 did not. How many standard deviations is this poll result away from the aide's model of reality? [15 pts.]

2. [*Random Variables*] In a best-of-5 college baseball playoff series, team A has a probability $p$ of winning a game against team B. Assume each game is statistically independent and that $p$ does not change. The random variable, $X$, of interest is the number of games played when the playoff series is completed. (Hint: use $q = 1 - p$.)

   a. Define the random variable $X$ and derive its probability density function. [10 pts.]
   b. Compute the expected value of $X$. Do not simplify. [5 pts.]

3. [*Expectation/Variation*] The standard deviation, $\sigma = \sqrt{E[(X-\mu)^2]}$, measures a "typical" difference between a data value and the center, $\mu$. Statisticians in the first part of the 20th Century used a more intuitive definition of "spread" called the *mean absolute deviation*, which is defined by

$$\tau = E\Big[\big|X - \mu\big|\Big] ;$$

   note that $\big|X - \mu\big|$ is the absolute value of $X - \mu$, or the distance from $X$ to $\mu$.

   a. Compare $\sigma$ and $\tau$ if $X \sim U(0,1)$. [7 pts.]
   b. Compare $\sigma$ and $\tau$ if $X \sim N(0,1)$. [8 pts.]

4. [*Transformation of Random Variables*] Suppose $X$ is a Cauchy random variable

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty .$$

   a. Find the pdf, $g(y)$, of the random variable $Y = 1/X$. [12 pts.]
   b. Can you name this pdf? (Try to simplify your answer in part (a) as much as possible.) [3 pts.]

5. [*Random Number Generation*] On a computer, given as many $U(0,1)$ samples $\{u_1, u_2, u_3, \ldots\}$ as desired, we wish to find an explicit algorithm for generating points at random in the interior of a unit circle; that is, radius = 1, center = (0,0), and $f(x,y) = 1/\pi$ on the circle and 0 elsewhere.

   a. Following the Box-Müller polar-coordinates approach (that is, use the radial symmetry of the circle), find the formulas that show how two points $(u_1, u_2)$ can be used to generate the desired random point $(x,y)$ in polar coordinates, $(r, \theta)$. Also provide the transformation back to Euclidean coordinates $(x,y)$. [10 pts.]
   b. If the point $(u_3, u_4)$ is scaled to be $(2u_3 - 1, 2u_4 - 1)$, then we have a random point on the square $[-1,1] \times [-1,1]$. Can you describe how such random points can be used to provide an alternative solution to the problem in part (a)? [5 pts.]

6. [*Correlation Coefficient*] Prove that the correlation coefficient between random variables $X$ and $Y$ always falls between $-1$ and 1. (Hint: You may assume $\mu_x = \mu_y = 0$ and $\sigma_x = \sigma_y = 1$.) [15 pts.]

7. [*Conditional PDF's*] If the height of a father, $X$, and his adult son, $Y$, follow a bivariate normal density with parameters $\mu_x = 68.0''$, $\mu_y = 69.0''$, $\sigma_x = \sigma_y = 2.5''$, and $\rho_{xy} = 0.5$, then find

   a. the Prob$(Y \geq 72.0'')$; [5 pts.]
   b. the conditional density $f(y|X = x)$; [5 pts.]
   c. and the Prob$(Y \geq 72.0''|X = 72.0'')$. [5 pts.]
   d. At what father's height is there a 50-50 chance that the son's height is greater than or equal to $72.0''$? [5 pts.]

8. [*Correlation Coefficient*] A pair of random variables $(X,Y)$ has the density

$$f(x,y) = \begin{cases} \frac{20}{3}y & \text{for } x^2 \leq y \leq \sqrt{x} \text{ and } 0 \leq x \leq 1; \\ \text{zero} & \text{elsewhere,} \end{cases}$$

   which is approximately a leaf-shaped region within the square $[0,1] \times [0,1]$.

   a. Sketch the density $f(x,y)$ as a contour plot. [5 pts.]
   b. Find the marginal pdf's $f_1(x)$ and $f_2(y)$. [5 pts.]
   c. Compute the correlation coefficient given that

$$\mu_x = \frac{5}{9} \quad \sigma_x^2 = \frac{55}{1134} \quad \mu_y = \frac{4}{7} \quad \sigma_y^2 = \frac{58}{1323}\,. \qquad [5 \text{ pts.}]$$

*Good Luck.* *Sign Pledge*