Solutions Midterm Exam — Stat 410

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1. see web page
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4. We want to show that the point $(\bar{x}, \bar{y})$ is on the regression surface (hyperplane). Suppose X is an $n \times p$ data matrix (with the first column a vector of 1's for the intercept), and that Y is the $n \times 1$ vector of responses. The least-squares estimate of β is $\hat{\beta} = (X^T X)^{-1} X^T Y$.

   (a) Let the vector of ones be denoted by $1_n = (1, 1, \ldots, 1)^T$. Show that the $p \times 1$ vector $\bar{x}$ and the scalar $\bar{y}$ can be computed as

   $$\bar{x} = \frac{1}{n} X^T 1_n \qquad \text{and} \qquad \bar{y} = \frac{1}{n} 1_n^T Y \,.$$

   - check $\bar{x}_k = \frac{1}{n} \sum_{i=1}^n x_{ik}$ for all $k = 1, \ldots, p$

   (b) Recall the hat matrix $H = X(X^T X)^{-1} X^T$ is idempotent, and that the vector of predictions at the original data points is given by $\hat{Y} = HY$. Show that the vector 1_n is unchanged by H, that is,

   $$H\, 1_n = 1_n \,.$$

   Hint: Compute the matrix product HX.

   - Recall the first column of X is the vector 1_n. Now

     $$HX = \left[X(X^T X)^{-1} X^T\right] X = X \,,$$

     assuming X is of full rank. It follows that $H\, 1_n = 1_n$ by looking at the first column of X alone.

   (c) The linear prediction at $x = \bar{x}$ is given by

   $$\hat{y} = \bar{x}^T \hat{\beta} \,.$$

   Show that this $\hat{y}$ is exactly $\bar{y}$.

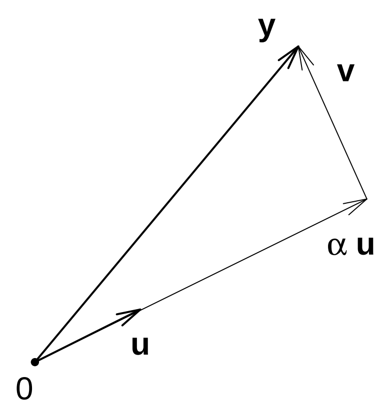
   - The prediction at a new point x_u is $\hat{y} = x_u^T \hat{\beta}$. Thus when $x_u = \bar{x}$,

     $$\hat{y} = \bar{x}^T \hat{\beta} = \left[\frac{1}{n} X^T 1_n\right]^T \left[(X^T X)^{-1} X^T Y\right]$$

     $$= \frac{1}{n}\, 1_n^T X (X^T X)^{-1} X^T Y = \frac{1}{n}\, 1_n^T\, HY = \frac{1}{n} 1_n^T Y = \bar{y} \,,$$

     since

     $$1_n^T H = 1_n^T H^T = (H\, 1_n)^T = 1_n^T \,.$$



A well-known identity used to compute the sample variance of a set of data $y = (y_1, y_2, \ldots, y_n)^T$ is

$$\frac{1}{n}\sum_{i=1}^{n}(y-\bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \bar{y}^2 \,.$$

This result is easily shown by ordinary algebra, but we want to demonstrate this identity by using the Pythagorean Theorem in n-dimensions, $\Re^n$.

(a) In the figure above, the vector $\mathbf{u}$ is a vector of length one in the direction $1_n = (1, 1, \ldots, 1)^T$. Find $\mathbf{u}$ in terms of the vector 1_n so that $\mathbf{u}^T\mathbf{u} = 1$.

- "In the direction" implies

$$\mathbf{u} = c\,1_n$$

and thus

$$1 = \mathbf{u}^T\mathbf{u} = (c\,1_n)^T(c\,1_n) = c^2\,1_n^T 1_n = c^2\,n\,.$$

Thus

$$c^2 = \frac{1}{n} \qquad \Rightarrow \qquad \mathbf{u} = \frac{1}{\sqrt{n}}1_n$$

(b) The vector $\mathbf{y}$ can be written as the sum of the two perpendicular vectors

$$\mathbf{y} = \alpha\,\mathbf{u} + \mathbf{v}\,,$$

where $\alpha\,\mathbf{u}$ is a vector in the same direction as $\mathbf{u}$. Note that the length of the vector $\alpha\,\mathbf{u}$ is $|\alpha|$, where α is a scalar. Recall vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u}^T\mathbf{v} = 0$. Find the unique value of the scalar, α, that makes $\alpha\,\mathbf{u}$ and $\mathbf{v}$ perpendicular by multiplying both sides of $\mathbf{y} = \alpha\,\mathbf{u} + \mathbf{v}$ by the vector $\mathbf{u}^T$; solve for α.

- Computing,

$$\mathbf{u}^T\mathbf{y} = \mathbf{u}^T\left[\alpha\,\mathbf{u} + \mathbf{v}\right] = \alpha\,\mathbf{u}^T\mathbf{u} + \mathbf{u}^T\mathbf{v} = \alpha\,1 + 0 = \alpha\,.$$

- Looking ahead, let us find an expression for α. Substituting for $\mathbf{u}$, we find

$$\alpha = \mathbf{u}^T\mathbf{y} = \left(\frac{1}{\sqrt{n}}\,1_n\right)^T\mathbf{y} = \frac{1}{\sqrt{n}}\,1_n^T\,\mathbf{y}\,.$$

  Now $\bar{y}$ is hiding in there:

$$\alpha = \frac{1}{\sqrt{n}}\,(n\,\bar{y}) = \sqrt{n}\,\bar{y}\,.$$

(c) Compute $\mathbf{v} = \mathbf{y} - \alpha\mathbf{u}$. Show that the Pythagorean Theorem with these 3 vectors proves the variance identity. Note the Pythagorean Theorem states that

$$||\mathbf{y}||^2 = ||\alpha\mathbf{u}||^2 + ||\mathbf{v}||^2$$

or

$$\mathbf{y}^T\mathbf{y} = (\alpha\mathbf{u})^T(\alpha\mathbf{u}) + \mathbf{v}^T\mathbf{v}\,.$$

- We are almost there. To find an expression for $\mathbf{v}$ we first simplify

$$\alpha\,\mathbf{u} = \left(\sqrt{n}\,\bar{y}\right)\cdot\left(\frac{1}{\sqrt{n}}\,1_n\right) = \bar{y}\,1_n\,.$$

Therefore,

$$\mathbf{v} = \mathbf{y} - \alpha\,\mathbf{u} = \mathbf{y} - \bar{y}\,1_n\,.$$

Finally,

$$||\mathbf{y}||^2 = \mathbf{y}^T\mathbf{y} = \sum_{i=1}^{n} y_i^2\,,$$

$$||\alpha\mathbf{u}||^2 = \alpha^2 = \left(\sqrt{n}\,\bar{y}\right)^2 = n\,\bar{y}^2$$

and

$$\mathbf{v}^T\mathbf{v} = \sum_{i=1}^{n}(y_i - \bar{y})^2$$

since

$$\mathbf{y} - \bar{y}\,1_n = \begin{pmatrix} y_1 - \bar{y} & y_2 - \bar{y} & \cdots & y_n - \bar{y}\end{pmatrix}^T\,.$$

This proves the identity (after dividing everything by n).