

Some Notes on Brownian Motion and Its Relationship to Black-Scholes

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Part 1: From Random Walks to Brownian Motion

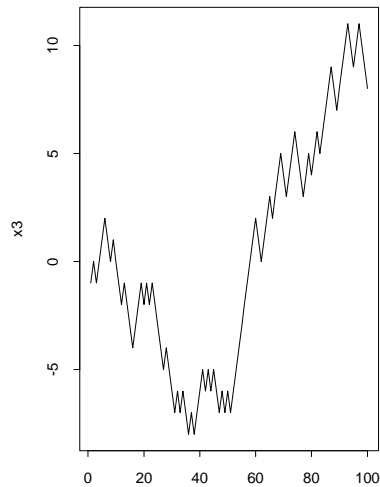
Define the Random Walk

Let X_1, X_2, \dots be independent and identically distributed (*iid*) random variables such that

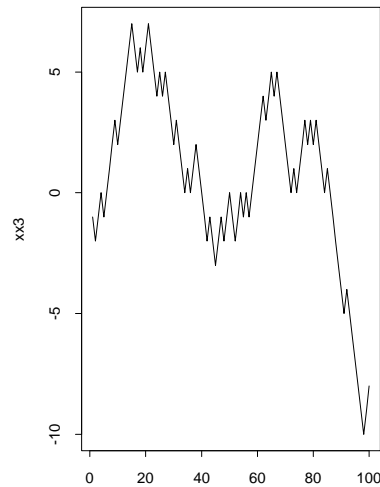
$$X_i = \begin{array}{ll} 1 & \text{probability} = \frac{1}{2} \\ -1 & \text{probability} = \frac{1}{2} \end{array}$$

Let B_n denote the filtration $\sigma(X_1, \dots, X_n)$ and let $S_n = \sum_{i=1}^n X_i$ where $X_0 = 0$ and hence $S_0 = 0$. When the X_i are defined as above with equal probabilities, S_n is referred to as a *symmetric simple random walk*.

Random Walk p=.5, seed=12345



Random Walk p=.5, seed=54321



Def: Let S_0, S_1, S_2, \dots be integrable random variables and let B_0, B_1, B_2, \dots be sigma fields such that $B_0 \subset B_1 \subset B_2 \subset \dots$ then $\{B_n\}$ is a *filtration*.

In the economics and finance literature a filtration is commonly referred to as the *information set*. The sigma field $\sigma(X_1, \dots, X_n)$ contains all the information about the evolution of the random variable S_n .

Def: $\{S_n\}$ is a *martingale* with respect to the filtration $\{B_n\}$ if

1. S_n is measurable with respect to $\{B_n\}$ for all n

2. $E[S_{n+1}|B_n] = S_n$ almost surely

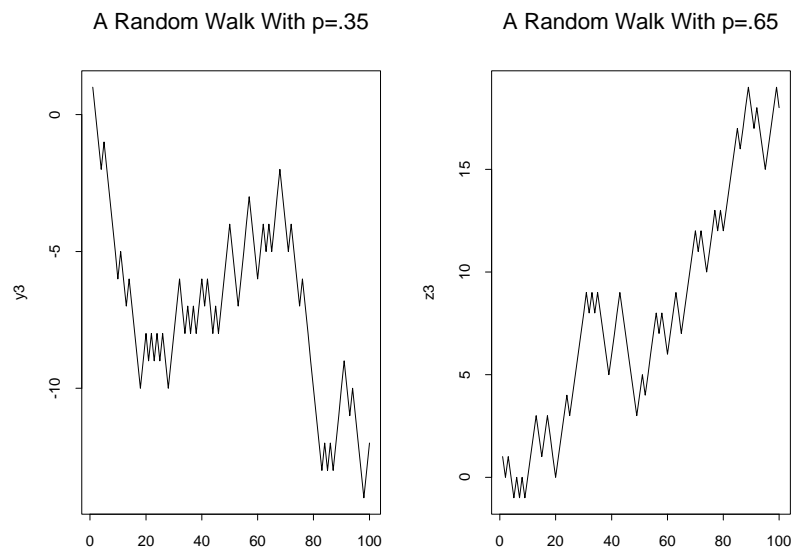
Our random walk $S_n = \sum_{i=1}^n X_i$ is a martingale.

For a more general random variable

$$X_i = \begin{cases} 1 & \text{probability} = p \\ -1 & \text{probability} = 1 - p \end{cases}$$

Def: If $p \geq \frac{1}{2}$ then $E[S_{n+1}|B_n] \geq S_n$ and S_{n+1} is a *submartingale*. Submartingales tend towards $+\infty$.

Def: If $p \leq \frac{1}{2}$ then $E[S_{n+1}|B_n] \leq S_n$ and S_{n+1} is a *supermartingale*. Supermartingales tend towards $-\infty$.



Use the Intuition of the Random Walk to Understand Brownian Motion

Think of a Brownian motion process as the limit of a sped up random walk, a random walk with an infinitesimal step. Let the random walk take a step of size Δx for every unit of time Δt , then

$$X(t) = \Delta x (X_1 + \dots + X_n) \text{ where } n = \frac{t}{\Delta t}$$

and

$$X_i \stackrel{iid}{\sim} P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Since

$$E[X_i] = 0,$$

$$E[X(t)] = E[\Delta x \sum_i X_i] = 0$$

and

$$\begin{aligned} \text{Var} [X (t)] &= \text{Var} [\Delta x \sum_i X_i] = (\Delta x)^2 \sum_i \text{Var} (X_i) = \\ &= (\Delta x)^2 \sum_i E (X_i^2) = (\Delta x)^2 \sum_i 1 \\ &= (\Delta x)^2 n = (\Delta x)^2 \left(\frac{t}{\Delta t} \right) \end{aligned}$$

If we want to motivate the Brownian motion as the limit of a random walk, we will need $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$ in a meaningful manner. If we let $\Delta x = c\sqrt{\Delta t}$, then

$$\text{Var} [X (t)] = c^2 (\Delta t) \left(\frac{t}{\Delta t} \right) = c^2 t$$

Since the steps of the random walk, the random variables X_i , are independent, application of the Central Limit Theorem would suggest that

$X = \Delta x (X_1 + \dots + X_n)$ is distributed as $N(0, c^2 t)$.

By the CLT

$$\frac{X(t) - E[X(t)]}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1),$$

$$\frac{X(t) - E[X(t)]}{\frac{\sigma}{\sqrt{n}}} = \frac{X(t) - 0}{\frac{\Delta x \left(\frac{t}{\Delta t}\right)}{\left(\frac{t}{\Delta t}\right)}} = \frac{X(t)}{\Delta x} = \frac{X(t)}{c\sqrt{t}} \sim N(0, 1)$$

$$X(t) \sim N(0, c^2 t)$$

The property of independence between non-overlapping time intervals in the random walk, suggests that the Brownian motion $\{X(t), t \geq 0\}$ should also have independent increments.

Def: The stochastic process $X(t), t \in T$ is said to have *independent increments* if for every choice of $t_i \in T$ with $t_1 < \dots < t_n$ and $n \geq 1$, $X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$ are independent random variables.

Since the distribution of the change in position of the random walk depends only on the length of the time interval, this suggests that the Brownian motion $\{X(t), t \geq 0\}$ has stationary increments.

Def: The stochastic process $X(t), t \in T$ is said to have *stationary increments* if $X(t) - X(s) \stackrel{d}{=} X(t+h) - X(s+h) \forall t, s \in T, h \ni t+h, s+h \in T$

.

Properties apparent in the discrete case should carry over to the continuous case as the random walk becomes infinitesimal in time and step.

Defining the Brownian Motion Process

Brownian motion is named after the English botanist Robert Brown who observed the movement of pollen grains suspended in water in 1827. The mathematical theory of Brownian motion was first developed by Albert Einstein in 1905. However, the mathematical properties of the process were rigorously developed by MIT mathematician Norbert Wiener in 1918 so the process is also commonly referred to as the Wiener process. Electrical engineers will typically refer to the process as a Wiener process while physicists and mathematicians will use the term Brownian motion.

Def: A stochastic process $\{X(t), t \geq 0\}$ is a *Brownian motion process* if

1. $X(0) = 0$

2. $\{X(t), t \geq 0\}$ has stationary independent increments

3. For all $t > 0$, $X(t)$ is normally distributed with mean 0 and variance c^2t

When $c = 1$, the process is called a *standard Brownian motion*.

1. Notice that the variance grows as a function of time.
2. With probability 1, $X(t)$ is a continuous function of t .
3. Continuity implies that $X(t)$ evolves in time without jumps.
4. With probability 1, $X(t)$ is nowhere differentiable.

$X(t)$ is everywhere continuous but nowhere differentiable. $X(t)$ is not a smooth function. Recall the classic example from freshman calculus of a function that is not differentiable at a point- the saw tooth function. $X(t)$ is everywhere jagged without a unique tangent line at any point.

Fact: The property of independent increments implies that $X(t)$ is a Markov process.

$$\begin{aligned} &P(X(s+t) \leq a | X(s) = x, X(u), 0 \leq u < s) \\ &= P(X(s+t) - X(s) \leq a - x | X(s) = x, X(u), 0 \leq u < s) \\ &= P(X(s+t) - X(s) \leq a - x) \\ &= P(X(s+t) \leq a | X(s) = x) \end{aligned}$$

Intuitively, the Markov property states that where the process is in the next step depends on the current location but not on any previous past location of the process.

Fact: The density of the process $X(t) \sim N(0, t)$ is

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-x^2}{2t}\right)$$

Fact: The property of stationary, independent increments implies that the joint density has the form

$$f(x_1, x_2, \dots, x_n) = f_t(x_1) f_{t_2-t_1}(x_2 - x_1) \cdots f_{t_n-t_{n-1}}(x_n - x_{n-1})$$

Hence the conditional distribution of $X(s)$ given $X(t) = B$ where $s < t$ can be easily computed as

$$f_{s|t}(x|B) = \frac{f_s(x) f_{t-s}(B-x)}{f_t(B)} = K \exp\left(\frac{-t\left(x - \frac{s}{t}B\right)^2}{2s(t-s)}\right)$$

which is a normal density with mean and variance for $s < t$

$$E[X(s) | X(t) = B] = \frac{s}{t}B$$

$$\text{Var}[X(s) | X(t) = B] = \frac{s}{t}(t-s)$$

Def: A stochastic process $\{X(t), t \geq 0\}$ is a *Brownian motion process with drift coefficient μ* if

1. $X(0) = 0$
2. $\{X(t), t \geq 0\}$ has stationary independent increments
3. For all $t > 0$, $X(t)$ is normally distributed with mean μt and variance t

Def: If the stochastic process $\{X(t), t \geq 0\}$ is a Brownian motion, then the stochastic process $\{Y(t), t \geq 0\}$ defined by $Y(t) = \exp(X(t))$ is called a *geometric Brownian motion*.

Application of Geometric Brownian Motion

Recall that geometric Brownian motion is useful in modeling independent and identically distributed percentage changes. If $Y(t_n)$ is the price of a stock, then we can define $\frac{Y(t_n)}{Y(t_{n-1})}$ as a geometric Brownian motion. If we define

$$X(t_n) = \frac{Y(t_n)}{Y(t_{n-1})} \text{ with } Y(0) = 1 \text{ then}$$

$$X(t_{n-1}) = \frac{Y(t_{n-1})}{Y(t_{n-2})}, X(t_{n-2}) = \frac{Y(t_{n-2})}{Y(t_{n-3})}, \dots$$

$$Y(t_n) = X(t_1) X(t_2) \cdots X(t_n)$$

$$\log(Y(t_n)) = \sum \log(X(t_i))$$

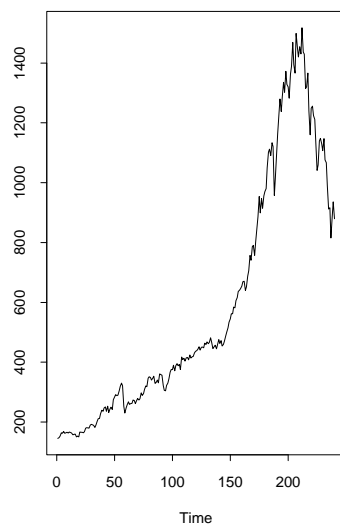
With the $X(t_i)$ being defined as iid, then $\log(Y(t_n))$ is approximately Brownian motion and hence $Y(t_n)$ is approximately geometric Brownian motion.

Part 2: Empirical Examples

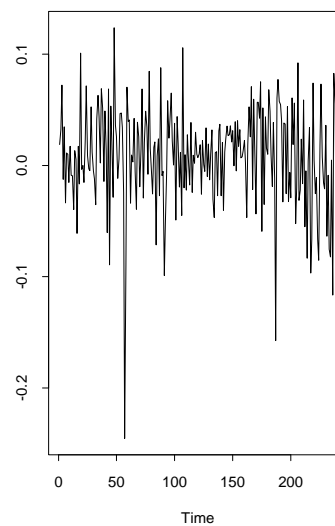
A Look at the S&P 500

Monthly Data of the S&P 500 from January 1983 to December 2002

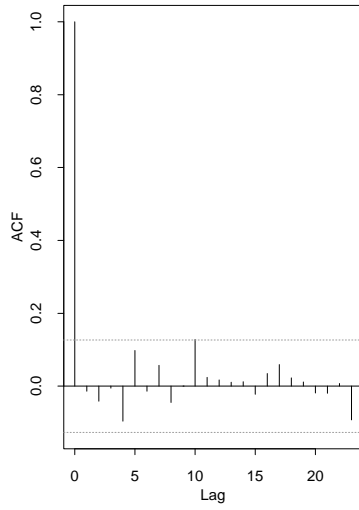
S&P 500 Monthly Price 1/83 to 12/02



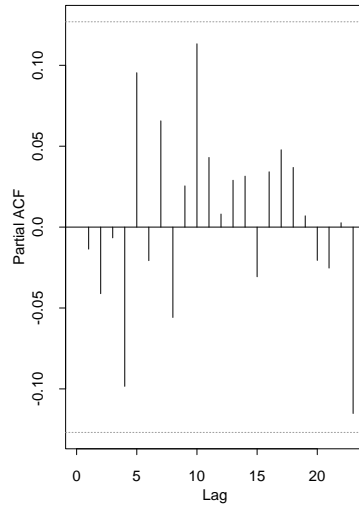
Monthly Log Returns



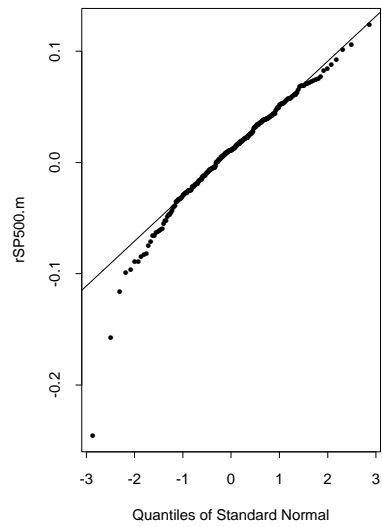
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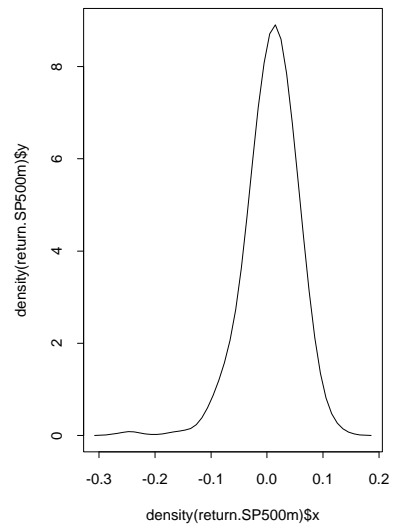
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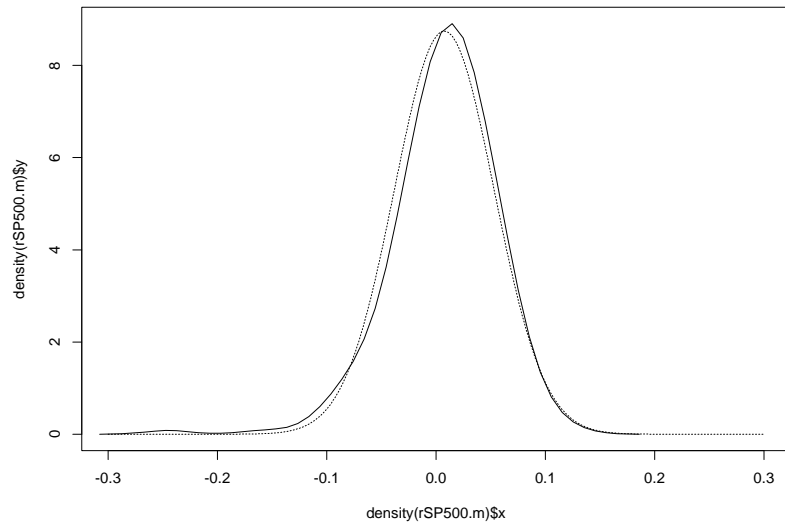
Q-Q Plot of Daily Returns



Density Estimate of Daily Returns

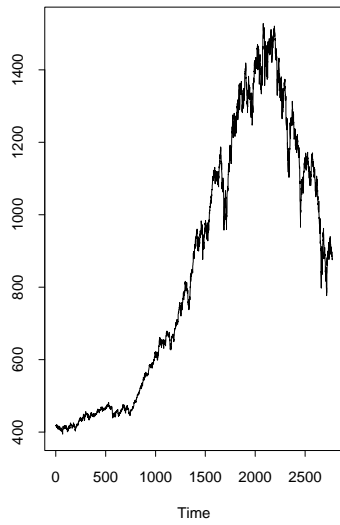


Density Comparison

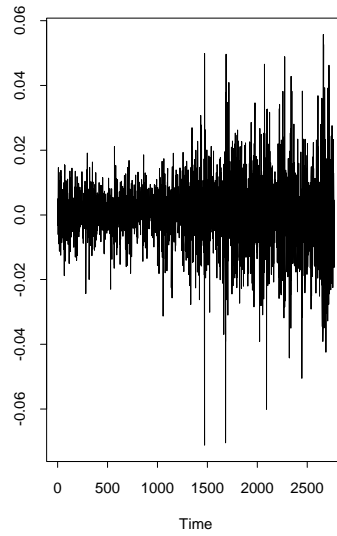


Daily Data of the S&P 500 from 1/1/1992 to 12/31/2002

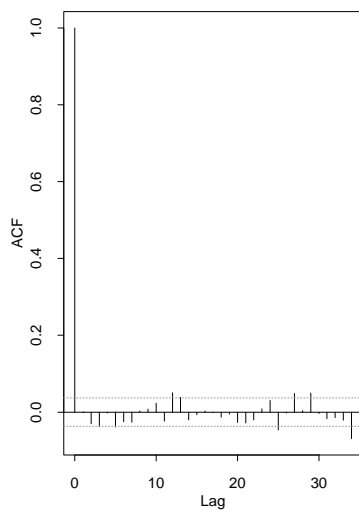
S&P 500 Daily Price 1992-2002



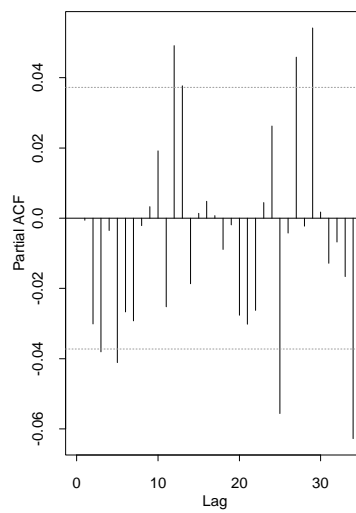
Daily Log Returns



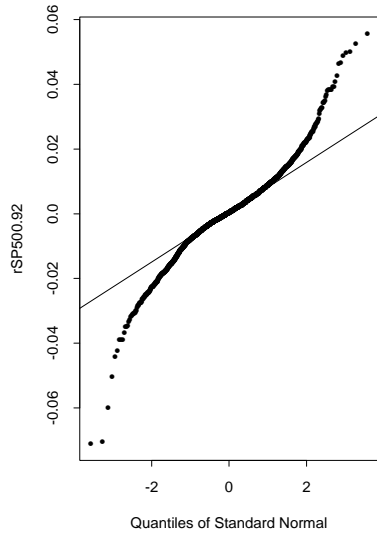
Series : rSP500.92



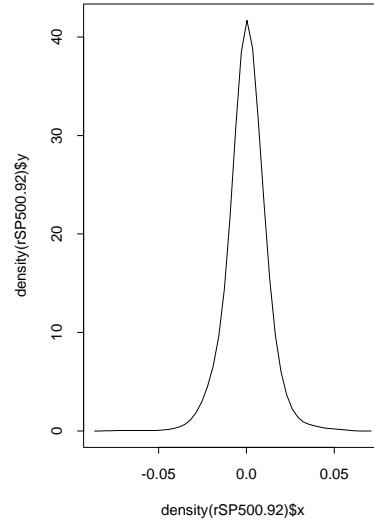
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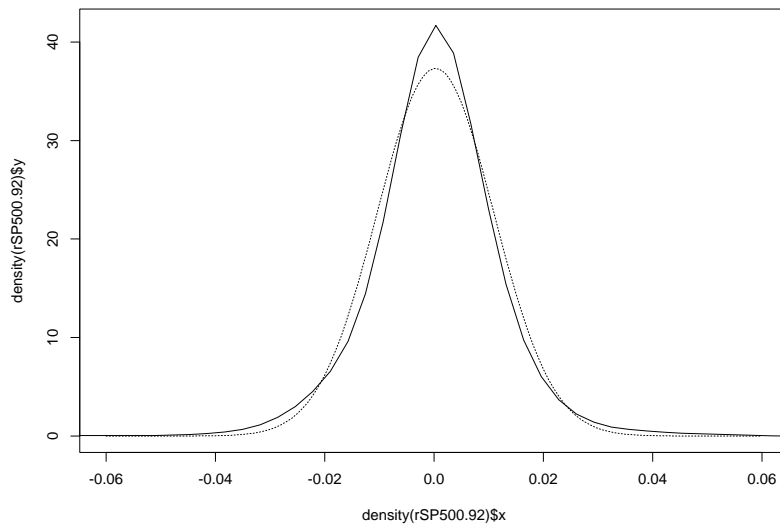
Q-Q Plot of Daily Returns



Density Estimate Daily Returns

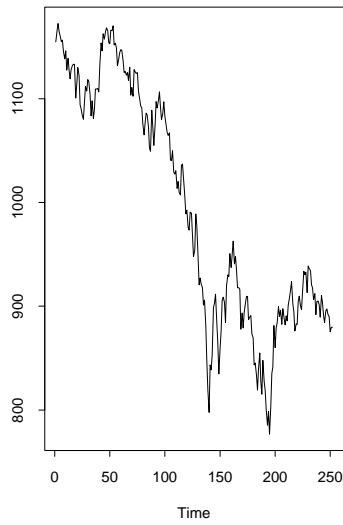


Density Comparison

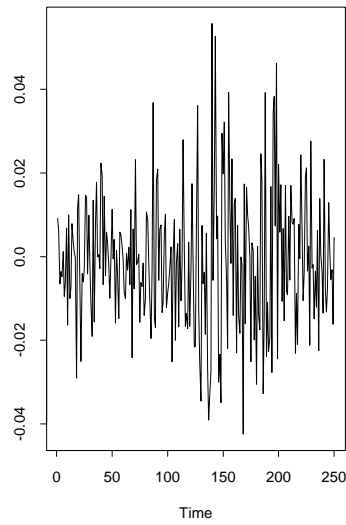


Daily Data of the S&P 500 from 1/1/2002 to 12/13/2002

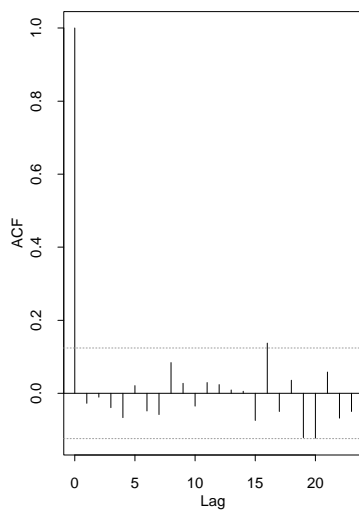
S&P 500 Daily Price 2002



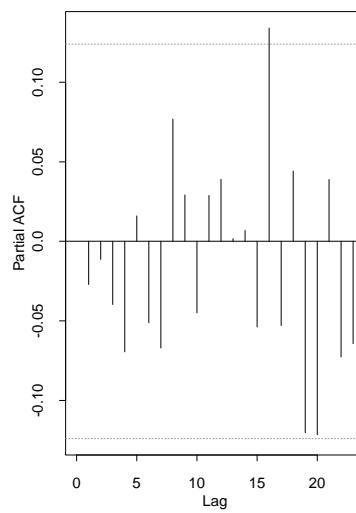
Daily Log Returns



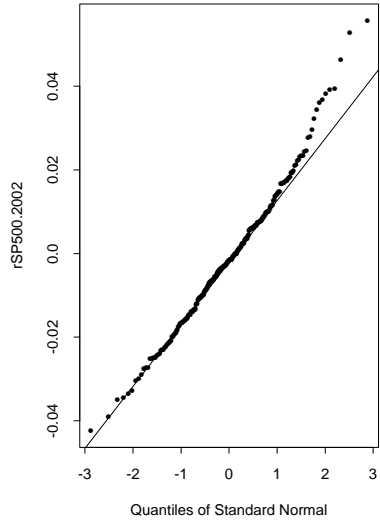
Series : rSP500.2002



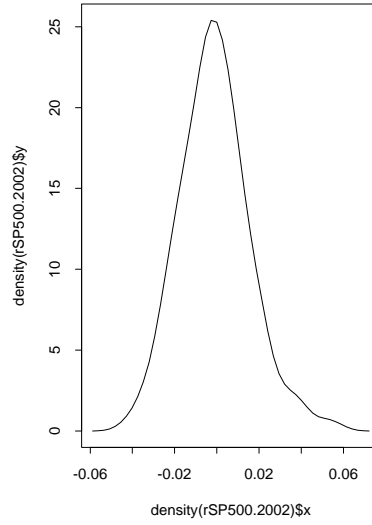
Series : rSP500.2002



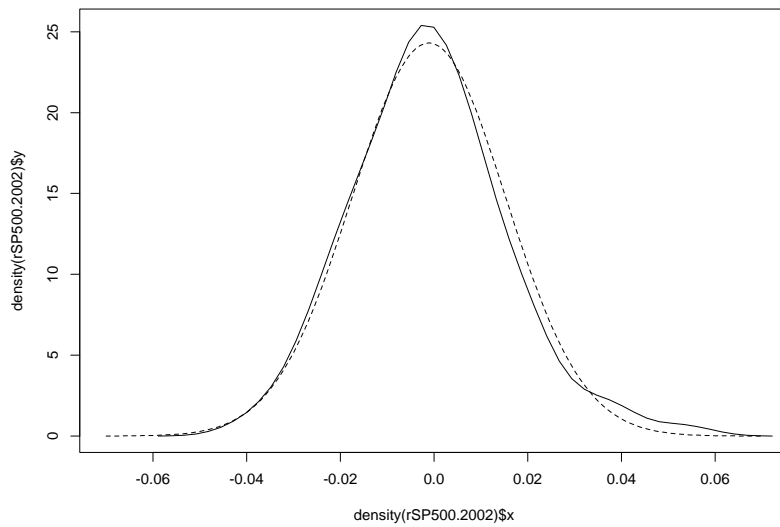
Q-Q Plot of Daily Returns



Density Estimate of Daily Returns

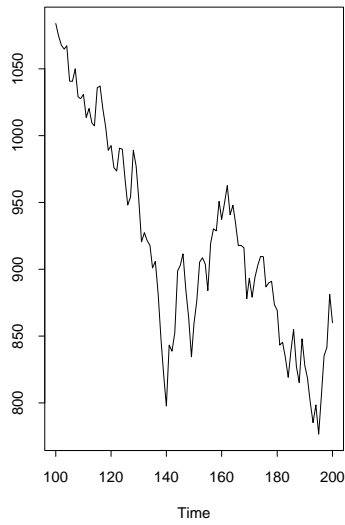


Density Comparison

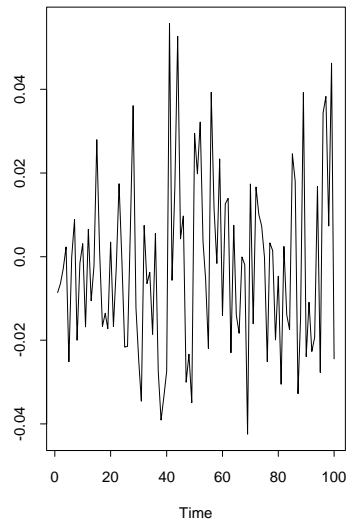


Trading Days 100 to 200 of S&P 500 data from 2002

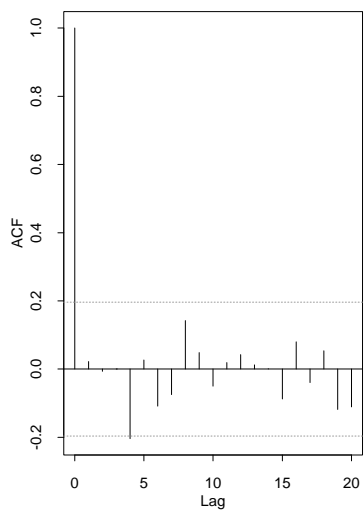
S&P 500 Daily Price



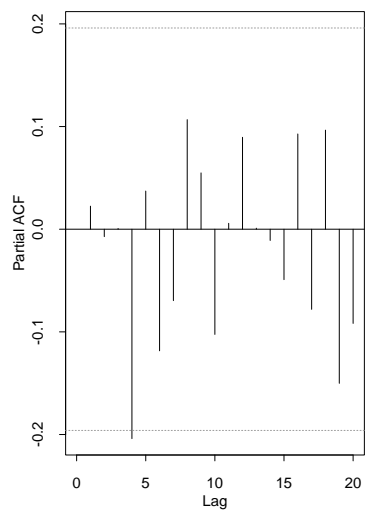
Daily Log Returns



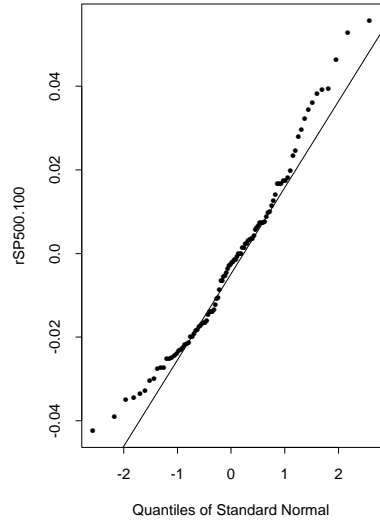
Series : rSP500.100



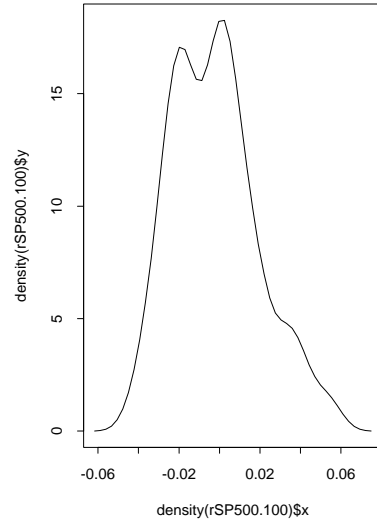
Series : rSP500.100



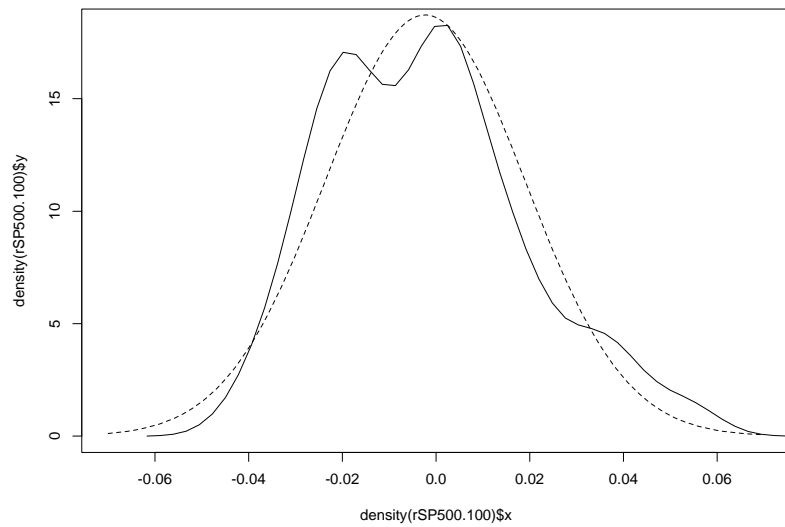
Q-Q Plot of Daily Returns



Density Estimate Daily Returns



Density Comparison



Summary Table of the Moments for the Data Sets

	Monthly	1992-2002	2002	100 days
Mean	0.0075	0.0002	-0.0010	-0.0023
Variance	0.0020	0.0001	0.0002	0.0004
Skewness	-1.0480	-0.1244	0.4312	0.5508
Kurtosis	3.9078	4.0727	0.6776	-0.0625

The student interested in the price process is referred to

1. Bachelier, Louis (1900). "Theorie de la Speculation." in P. H. Cootner (Ed.) (1964), *The Random Character of Stock Market Prices*, 17-78. Cambridge, MA: MIT Press.
2. Fama, E. F., "The Behavior of Stock Prices," *Journal of Business*, 38 (January 1965), 34-105.
3. Kon, S. J., "Models of Stock Returns- A Comparison," *Journal of Finance*, 39 (March 1984), 147-65.

4. Osbourne, M. F. M. (1958). "Brownian Motion in the Stock Market." in P. H. Cootner (Ed.) (1964), *The Random Character of Stock Market Prices*, 100-128. Cambridge, MA: MIT Press.

5. Samuelson, P. (1965). "Proof That Properly Anticipated Prices Fluctuate Randomly," *Industrial Management Review*, (Spring 65), Vol. 6, Issue 2, 41-50.

Part 3: From Stochastic Processes to Stochastic Differential Equations

Recall that a Brownian motion path is continuous almost everywhere but nowhere differentiable. Hence for a standard Brownian motion ω_t , $\frac{d\omega_t}{dt}$ does not exist for all elements of Ω except for elements in a subset $\Omega_1 \subset \Omega$ such that $P(\Omega_1) = 0$. One needs a different type of integration than the standard Riemann or Lebesgue integration theories to handle integrals involving Brownian motion - Ito calculus.

These notes have previously introduced the Wiener process or Brownian motion as the limit of a random walk, but one can more easily express the process as a stochastic differential equation (*SDE*). Note, both differential form (continuous) and difference form (discrete) will be used interchangeably.

The Generalized Wiener Process

Def: The continuous and discrete forms for the *generalized Wiener process* are

$$dx_t = \mu dt + \sigma dw_t$$

$$x_t - x_0 = \Delta x = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}, \quad \varepsilon \sim N(0, 1)$$

The parameters μ and σ are referred to as the *drift* and *volatility* of the generalized Wiener process x_t . In the generalized Wiener process the drift and volatility parameters are defined to be time invariant.

The Ito Process and Ito's Lemma

If one allows μ and σ from the generalized Wiener process to be functions of the stochastic process x_t such that $\mu(x_t, t)$ and $\sigma(x_t, t)$, the process becomes a stochastic diffusion called the *Ito process*. The *SDE* for the Ito process is given by

$$dx_t = \mu(x_t, t) dt + \sigma(x_t, t) dw_t$$

Another representation of the Ito process is by the integral equation

$$x_t = x_0 + \int_0^t \mu(x_s, s) ds + \int_0^t \sigma(x_s, s) dw_s$$

Be sure to recognize that the second integral in the equation is a stochastic integral. The Wiener process is a special case of an Ito process with $\mu(x_t, t) = 0$ and $\sigma(x_t, t) = 1$.

Def: Ito's Lemma

Assume x_t is a continuous-time stochastic process satisfying the *SDE*

$$dx_t = \mu(x_t, t) dt + \sigma(x_t, t) dw_t$$

where w_t is a Wiener process. Let $G(x_t, t)$ be a differentiable function of x_t and t , then

$$dG = \left(\frac{\partial G}{\partial x} \mu(x_t, t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma^2(x_t, t) \right) dt + \frac{\partial G}{\partial x} \sigma(x_t, t) dw_t$$

PROOF: Proof follows by a second order Taylor series expansion.

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (dx)^2 + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (dt)^2 + \frac{1}{2} \frac{\partial^2 G}{\partial x \partial t} dx dt$$

Letting $dw = \sqrt{dt}$ and substituting the Ito process into the Taylor expansion we get

$$dx = \mu(x, t) dt + \sigma(x, t) \sqrt{dt}$$

$$dxdt = \mu(x, t) dt^{\frac{3}{2}} + \sigma(x, t) dt^{\frac{3}{2}} = o\left(dt^{\frac{3}{2}}\right)$$

$$(dx)^2 = (\sigma(x, t))^2 dt + o\left(dt^{\frac{3}{2}}\right)$$

Substituting into G we get

$$dG = \frac{\partial G}{\partial x} (\mu(x, t) dt + \sigma(x, t) \sqrt{dt}) + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} ((\sigma(x, t))^2 dt + o(dt)) \\ + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} o(dt^{\frac{3}{2}}) + \frac{1}{2} \frac{\partial^2 G}{\partial x \partial t} o(dt^{\frac{3}{2}})$$

Letting the $o\left(dt^{\frac{3}{2}}\right)$ terms go to zero, one is left with

$$dG = \frac{\partial G}{\partial x} (\mu(x, t) dt + \sigma(x, t) \sqrt{dt}) + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\sigma(x, t))^2 dt$$

or in a more convenient form

$$dG = \left(\frac{\partial G}{\partial x} \mu(x_t, t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma(x_t, t) \right) dt + \frac{\partial G}{\partial x} \sigma(x_t, t) dw_t$$

Application of Ito's Lemma

Let P_t be the price of a stock at time t , which is continuous in $[0, \infty)$. Assume P_t follows the special Ito's process

$$dP_t = \mu P_t dt + \sigma P_t dw_t$$

where μ and σ are constant. From the general form of an Ito process

$$dx = \mu(x, t) dt + \sigma(x, t) \sqrt{dt}$$

one can see that $\mu(x, t) = \mu P_t$ and $\sigma(x, t) = \sigma P_t$.

Apply Ito's lemma to obtain a continuous-time model for the logarithm of the stock price P_t . Let $G(P_t, t) = \ln(P_t)$. From Ito's lemma it is known that

$$dG = \left(\frac{\partial G}{\partial x} \mu(x_t, t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma^2(x_t, t) \right) dt + \frac{\partial G}{\partial x} \sigma(x_t, t) dw_t$$

$$\text{where } \frac{\partial G}{\partial P_t} = \frac{1}{P_t}$$

$$\frac{\partial G}{\partial t} = 0$$

$$\frac{\partial^2 G}{\partial P_t^2} = \frac{-1}{P_t^2}$$

$$d \ln(P_t) = \left(\frac{1}{P_t} \mu P_t + 0 + \frac{1}{2} \frac{-1}{P_t^2} \sigma^2 P_t^2 \right) dt + \frac{1}{P_t} \sigma P_t dw_t$$

$$d \ln(P_t) = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dw_t$$

which is a generalized Wiener process with drift rate $\left(\mu - \frac{1}{2} \sigma^2 \right)$ and variance rate σ^2 .

Hence

$$\ln(P_T) - \ln(P_t) \sim N\left(\left(\mu - \frac{1}{2}\sigma\right)(T - t), \sigma^2(T - t)\right)$$

Hence this model suggests that log returns follow a geometric Brownian motion.

Derivation of the Black-Scholes Differential Equation

Putting the pieces together one can show how to develop the Black-Scholes model from the assumption of a geometric Brownian motion for the return process. Assume that the stock price P_t follows a geometric Brownian motion

$$dP_t = \mu P_t dt + \sigma P_t dw_t \quad \text{EQ(1)}$$

and let $G_t = G(P_t, t)$ be the price of a derivative contingent on P_t . Applying Ito's lemma yields

$$dG = \left(\frac{\partial G}{\partial x} \mu(x, t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma^2(x, t) \right) dt + \frac{\partial G}{\partial x} \sigma(x, t) dw_t$$

In this particular case $\mu(x, t) = \mu P_t$ and $\sigma(x, t) = \sigma P_t$ so dG becomes

$$dG_t = \left(\frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) dt + \frac{\partial G_t}{\partial P_t} \sigma P_t dw_t \quad \text{EQ(2)}$$

Notice both EQ(1) and EQ(2) contain the Wiener process. However, one may construct a portfolio that does not involve the Wiener process by shorting the derivative G_t and going long on $\frac{\partial G_t}{\partial P_t}$ shares of stock. Denote the value of the portfolio by V_t . The constructed portfolio is then

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t \quad \text{EQ(3)}$$

and the rate of change in the value of the portfolio is

$$dV_t = -dG_t + \frac{\partial G_t}{\partial P_t} dP_t \quad \text{EQ(4)}$$

Substituting EQ(1) and EQ(2) into EQ(4) will demonstrate the beauty (mathematical tractability) of this formulation.

$$dV_t = - \left[\left(\frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) dt + \frac{\partial G_t}{\partial P_t} \sigma P_t dw_t + \frac{\partial G_t}{\partial P_t} [\mu P_t dt + \sigma P_t dw_t] \right]$$

$$dV_t = \left[-\frac{\partial G_t}{\partial t} - \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right] dt \quad \text{EQ(5)}$$

The resulting equation for the rate of change in the portfolio value no longer contains the Wiener process. Two stochastic equations yielded a deterministic equation. Even more importantly is that the drift term, μ , has been removed from the equation allowing the valuation to be risk-neutral. Now one must apply an equilibrium condition. The Black-Scholes solution is a general equilibrium solution. Solutions to models in physics and economics described by differential equations are often solved by imposing equilibrium conditions. The equilibrium condition we will use is called the *no arbitrage assumption*. Since the portfolio is long on the underlying asset and short the derivative, it consists of a hedged position at time t . This hedged position is riskless for the small time interval dt , and thus must earn the risk

free rate of return for that time interval. Hence the no arbitrage assumption implies

$$dV_t = rV_t dt \quad \text{EQ(6)}$$

By substituting EQ(3) and EQ(5) into EQ(6) one obtains

$$\left[-\frac{\partial G_t}{\partial t} - \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right] dt = r \left[-G_t + \frac{\partial G_t}{\partial P_t} P_t \right] dt$$

which simplifies to

$$\frac{\partial G_t}{\partial t} + rP_t \frac{\partial G_t}{\partial P_t} + \frac{1}{2} \sigma^2 P_t^2 \frac{\partial^2 G_t}{\partial P_t^2} = rG_t \quad \text{EQ(7)}$$

the Black-Scholes differential equation.

Exercise: For the Black-Scholes differential equation

$$\frac{\partial f_t}{\partial t} + rS_t \frac{\partial f_t}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f_t}{\partial S_t^2} = r f_t$$

show that

$$f(S, t) = S \Phi \left(\frac{\log\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right) - X \exp(-r(T-t)) \Phi \left(\frac{\log\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right)$$

satisfies the differential equation.

Assumptions in the Black-Scholes Model

The assumptions used in the derivation of the Black-Scholes differential equation

1. The stock price follows a geometric Brownian motion and hence μ and σ are constant and the price process evolves continuously without jumps.
2. Short sales of securities are allowed to generate a riskless portfolio.
3. No transaction cost or taxes are incurred.
4. All securities are perfectly divisible.
5. The underlying asset does not pay dividends. (The timing of dividend payments affects the stock price.)

6. The risk-free rate of interest, r , is constant and the same for all maturities.

Many of these assumptions have been relaxed in the option pricing models that have followed since the Black-Scholes model.

Part 4: The Effectiveness of Option Pricing Models

A variety of option pricing models exist. Several models have been proposed to remedy the deficiencies of the Black-Scholes model. In the end the effectiveness of an option pricing model is determined by its ability to develop profitable trading strategies. An option pricing model is an estimator. The estimator needs to be empirically validated and have its biases documented. The difference between theoretical option valuation and the market valuation is analogous to the difference between physics and engineering. All models will need adjustment in the wind tunnel.

For more on market efficiency and the effectiveness of option pricing models the student is referred to

1. Black, F., and M. Scholes, "The Valuation of Option Contracts and a Test of Market Efficiency," *Journal of Finance*, 27 (May 1972), 399-418.
2. Rubinstein, M., "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976 through August 31, 1978," *Journal of Finance*, 40 (June 1985), 455-80.

Part 5: Suggested Readings for the Interested Student

1. Bjork, T. (1998). *Arbitrage Theory in Continuous Time*. Oxford University Press, Oxford.
2. Duffie, D. (1992). *Dynamic Asset Pricing Theory*. Princeton University Press, Princeton, NJ.
3. Huang C. and R. H. Litzenberger (1988). *Foundations for Financial Economics*, North-Holland, New York.
4. Hull, J. C. (2003). *Options, Futures, and Other Derivative Securities*. Fifth Edition. Prentice Hall, New York.

5. Kijima, M. (2003). *Stochastic Processes with Applications to Finance*. Chapman & Hall/CRC.
6. Pliska, S. R. (1997). *Introduction to Mathematical Finance: Discrete Time Models*. Blackwell, Cambridge.
7. Ross, S. M. (1996). *Stochastic Processes*. 2nd Ed. Wiley, New York.
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9. Tsay, R. S. (2002). *Analysis of Financial Time Series*. Wiley, New York.

10. Clewlow, L., and C. Strickland (2000). *Energy Derivatives: Pricing and Risk Management*, Lacima Group.

11. Eydeland, A. and K. Wolyniec (2003). *Energy and Power Risk Management: New Developments in Modeling, Pricing, and Hedging*. Wiley, New York.