Stat331 HW#11 Solutions (60')

6.59 (5')

Assuming normal,

$$T = \frac{\overline{X}}{S/\sqrt{n}} \approx 1, \qquad t_{\alpha/2,n-1} = 2.78$$

$$T < t_{\alpha/2,n-1}, \qquad accept\ H_0$$

6.64 (5')

$$H_0 : \mu_2 \le \mu_1 \ \ vs.\ \ H_1 : \mu_2 > \mu_1$$

$$X = X_2 - X_1, S = SD(X)$$

$$T = \frac{\overline{X}}{S/\sqrt{n}} = 1.69, \quad t_{\alpha,n-1} = 1.81$$

$$T < t_{\alpha,n-1}, \qquad accept\ H_0$$

6.66 (5')

$$p_0 = 21\%, \qquad H_0 : p < p_0 \ \ vs.\ \ H_1 : p \ge p_0$$

$$T = \frac{23\% - 21\%}{\sqrt{21\% * (1-21\%)/1000}} = 1.55, \quad t_{\alpha,999} = 1.64$$

$$T < t_{\alpha,999}, \qquad accept\ H_0$$

7.3 (5')

$$S = \{D, N\},\ P = \begin{bmatrix} 0.3 & 0.7 \\ 0.01 & 0.99 \end{bmatrix}$$

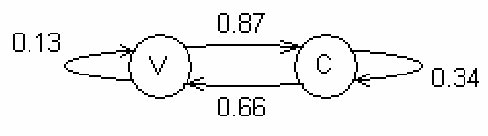
Limit distribution can be calculated by either linear system approach or by matrix approach, here gives the first one:

$$Solve \begin{cases} 0.3x_1 + 0.01x_2 = x_1 \\ x_1 + x_2 = 1 \end{cases}$$

$$\pi_1 = \frac{x_1}{x_1 + x_2} = \frac{1}{71}, \textit{which is the long term proportion of defective components}$$

7.9 (10')

(a)



$$P = \begin{bmatrix} 0.13 & 0.87 \\ 0.66 & 0.34 \end{bmatrix}$$

(b)

Either use $P^{(2)}$ *or compute directly*

$$P^{(2)} = P \times P = \begin{bmatrix} 0.59 & 0.41 \\ 0.31 & 0.69 \end{bmatrix}, \text{ so } P[X_3 = v \mid X_1 = v] = 0.59$$

$$P[X_3 = v \mid X_1 = v] = P[X_3 = v \mid X_2 = v]P[X_2 = v \mid X_1 = v] + P[X_3 = v \mid X_2 = c]P[X_3 = c \mid X_1 = v]$$
$$= 0.13^2 + 0.87 \times 0.66 = 0.59$$

(c)

Solve linear system to get limit distribution

$$\begin{cases} 0.13x_1 + 0.66x_2 = x_1 \\ x_1 + x_2 = 1 \end{cases} \Rightarrow \pi = \begin{bmatrix} 0.43 & 0.57 \end{bmatrix}$$

7.13 (5')

Proof:

$$\text{suppose } q \text{ is limit distribution}, q = \lim_{n \to \infty} p_{ij}^{(n)}$$

$$p_{ij}^{(n+1)} = \sum_{k \in S} p_{ik}^{(n)} p_{kj}$$

$$\Rightarrow \lim_{n \to \infty} p_{ij}^{(n+1)} = \lim_{n \to \infty} \sum_{k \in S} p_{ik}^{(n)} p_{kj}$$

$$\Rightarrow q_j = \sum_{k \in S} q_k p_{kj} \quad (\text{since } S \text{ finite, limit and } \Sigma \text{ can change order})$$

$$\Rightarrow q = q \cdot P$$

$$\Rightarrow q \text{ is statinary distribution}$$

Additional:

1. (5')

Proof: by induction

$(1)\, n = 1, X_1 = q_0 P$

$(2)\, \text{suppose } X_n = q_0 P^{(n)}$

$X_{n+1} = X_n P = q_0 P^{(n)} P = q_0 P^{(n+1)}$

If q_0 is a stationary distribution, then the MC is full stationary, i.e., $X_n = q_0, \forall n$.

2. (10')

(a)

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \text{ graph omitted.}$$

(b)

$$P^{(n)} \to \begin{bmatrix} 0.1724 & 0.2069 & 0.6207 \\ 0.1724 & 0.2069 & 0.6207 \\ 0.1724 & 0.2069 & 0.6207 \end{bmatrix}$$

Limit distribution exists. $\pi = [0.1724 \quad 0.2069 \quad 0.6207]$

(c)

The limit distribution $\pi = \begin{bmatrix} \frac{5}{29} & \frac{6}{29} & \frac{18}{29} \end{bmatrix} = [0.1724 \quad 0.2069 \quad 0.6207]$, agree with (b).

3. (10')

We can use a 9-state MC model, but a 3-state MC is simpler.

S={center, edge, corner}, in graph:

3	2	3
2	1	2
3	2	3

For king:

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/5 & 2/5 & 2/5 \\ 1/3 & 2/3 & 0 \end{bmatrix}$$

$\pi = [0.2 \quad 0.5 \quad 0.3]$

Note: to get limit distribution in 9 states, we need to divide by number of 1, 4 and 4.

6	2	7
5	1	3
9	4	8

$\pi = [0.2 \quad 0.125 \quad 0.125 \quad 0.125 \quad 0.125 \quad 0.075 \quad 0.075 \quad 0.075 \quad 0.075]$

For queen:

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/6 & 1/2 & 1/3 \\ 1/6 & 1/3 & 1/2 \end{bmatrix}$$

$\pi = [0.1429 \quad 0.4286 \quad 0.4286] = [1/7 \quad 3/7 \quad 3/7]$

For rook:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$\pi = [0.1111 \quad 0.4444 \quad 0.4444] = [1/9 \quad 4/9 \quad 4/9]$

Suppose center is white grid

For white bishop (ignore state 2)

$$P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\pi = [0.3333 \quad 0.6667] = [1/3 \quad 2/3]$$

For black bishop (ignore state 1 and 3)

$\pi = [1]$, always in edge grid.