Stat331 HW3 solution (40')

2.11 (2')
No. Example: Unif[0,1/2]
Yes. Consider the whole range $(-\infty,\infty)$

2.17 (5')
(a) $\int_0^x f(x)dx \geq 0$

$$\int_0^\infty f(x)dx = \int_0^1 \frac{1}{2}dx + \int_1^\infty \frac{1}{2x^2}dx = 1$$

(b) $F_X(x) = \begin{cases} \frac{x}{2}, 0 < x \leq 1 \\ 1 - \frac{1}{2x}, x > 1 \end{cases}$

(c) $f_Y(y) = F_y'(y) = f_X(x)$

2.28 (3')
The amount of money in 2 envelopes should be fixed and cannot be changed by which you choose.

2.30 (5')

$P[\text{the child chosen comes from a family with k children}]$

$$= \frac{(\text{\# of ways of choosing a family with k children}) \times (\text{\# of ways of choosing a child from that family})}{\text{\# of ways of choosing a child}}$$

$$= \frac{p_k \times k \times N}{\sum_{k=0}^{4} p_k \times k \times N}$$

$$= kp_k / \mu$$

can also be done by conditional probability.

2.31 (5')

Let $X = \text{max bid from others}, X \text{ random}$

$B = \text{money you bid}, B \text{ fixed}$

your profit is a function of $X, g(X) = \begin{cases} 0, & \text{if } B < X \\ 100 - B, & \text{if } B \geq X \end{cases}$

$$E[g(x)] = \int_{70}^{130} g(x)f(x)dx = \int_{70}^{B} \frac{100-B}{60}dx = \frac{1}{60}(100-B)(B-70)$$

when $B = 85$, $E[g(x)]$ reaches its maximum, 3.75

2.32 (5')

$Let\ X = length\ \ from\ starting\ \ end\ to\ broken\ \ point$

$\quad Y = length\ of\ the\ longest\ \ piece$

$$Y = \begin{cases} 1 - X, 0 < X \le 1/2 \\ X, \quad 1/2 < X \le 1 \end{cases}$$

$$E[Y] = \int_0^{1/2}(1-x)dx + \int_{1/2}^{1} x dx = 3/4$$

2.42 (5')

$\Leftarrow, easy$

$\Rightarrow, let\ c\sigma = 1/k$

$$P[|x-\mu| \ge c\sigma] \le \frac{1}{c^2} \Rightarrow P[|x-\mu| \ge 1/k] \le \sigma^2 k^2 \Rightarrow P[|x-\mu| \ge 1/k] = 0\ when\ \sigma = 0, \forall k \Rightarrow x \equiv \mu$$

2.82 (5')

(a) $F_X(x) = \int_{-\infty}^{x} f_X(x)dx = \frac{1}{2} + \frac{atan(x)}{\pi}$ (set x=tanθ to solve integral)

(b) $f_Y(y) = F_y'(y) = \frac{1}{\pi(1+y^2)} = f_X(x)$

Additional (5')

$$E[X] = \sum_{k=0}^{\infty} P[X>k] = \sum_{k=0}^{\infty}\sum_{x=k+1}^{\infty} p(x) = \frac{1}{2\sqrt{5}} \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\left[(\frac{a}{2})^{n-1} - (\frac{b}{2})^{n-1}\right] = \frac{1}{2\sqrt{5}}\left[\frac{1}{(1-\frac{a}{2})^2} - \frac{1}{(1-\frac{b}{2})^2}\right] = 6$$

$$or\ E[X] = \sum_{x=1}^{\infty} xp(x) = \frac{1}{\sqrt{5}}\sum_{x=1}^{\infty}\frac{x(a^{x-1}-b^{x-1})}{2^x} = \frac{1}{2\sqrt{5}}\sum_{x=1}^{\infty}\left[\frac{d}{d(\frac{a}{2})}(\frac{a}{2})^x - \frac{d}{d(\frac{b}{2})}(\frac{b}{2})^x\right]$$

$$= \frac{1}{2\sqrt{5}}\left[\frac{d}{d(\frac{a}{2})}\sum_{x=1}^{\infty}(\frac{a}{2})^x - \frac{d}{d(\frac{b}{2})}\sum_{x=1}^{\infty}(\frac{b}{2})^x\right] = \frac{1}{2\sqrt{5}}\left[\frac{1}{(1-\frac{a}{2})^2} - \frac{1}{(1-\frac{b}{2})^2}\right] = 6$$