Homework 4 – Book Problems

2.44 **(3 pt)**

Let X = # of 6's

X ~ Binomial(10, 1/6)

a) $P(X = 0) = \binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} = .161506$

b) $P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 0) - P(X=1) = 1 - .161506 - \binom{10}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9 = .51483$

c) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) = .930272$

2.48 **(7 pt)**

Let X = # of Correct Answers by Guessing

For part a and b: X ~ Binomial(6, ¼)

a) P(Pass by Guessing) =P(X=4) + P(X=5) + P(X=6) = .037598

b) P(Know 1st three, Guess on the Rest) = P(X=1) + P(X=2) +P(X=3)=37/64=.578125

c) and d) In the back of the book

e) (a) E(X) = 6(1/4) = 1.5 Var(X) = 6(1/4)(3/4) = 1.125
(b) E(X) = 3 + 3(1/4) = 3.75 Var(X) = 0 + 3(1/4)(3/4) = 9/16 = .5625
(c) E(X) = 6(5/8) = 3.75 Var(X) = 6(5/8)(3/8) = 1.40625

2.51 In the back of the book **(2 pt)**

Solutions to Additional Homework #4 Problems

1. Note that we have a Hypergeometric **(1 pt)**

Let X = Number of defectives in sample

$$P(X \geq 1) = 1 - P(X = 0)$$

$$P(X = 0) = \frac{\binom{5}{0}\binom{95}{10}}{\binom{100}{100}} = .583752$$

$$P(X \geq 1) = 1 - .583752 = .416248$$

2. Here we have a Poisson (λ=18) **(1 pt)**

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = .000018$$

3.

a) Given the identity and using the following representation of the Negative Binomial: **(2 pt)**

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, \ldots$$

$$\begin{aligned} M_x(t) &= \sum_{x=r}^{\infty} e^{tx} \binom{x-1}{r-1} p^r (1-p)^{x-r} \\ &= p^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} \left[(1-p)e^t\right]^x \left[(1-p)e^t\right]^{-r} e^{tr} \\ &= (pe^t)^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} \left[(1-p)e^t\right]^{x-r} \\ &= (pe^t)^r (1-(1-p)e^t)^{-r} \\ &= \left(\frac{pe^t}{1-(1-p)e^t}\right)^r \end{aligned}$$

NOTE: If you use $p(x) = \binom{r+x-1}{x} p^r (1-p)^x, x = 0, 1, 2\ldots$ .then $M_x(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$. The difference here is the use of the transformation Y = X + r.

b) **(2 pt)**

$$E(X) = M_X'(0) = \left[\frac{re^{tr}\left(\frac{p}{1-(1-p)e^t}\right)^r}{1-(1-p)e^t} \right]\Bigg|_{t=0} = \frac{r}{p}$$

$$E(X^2) = M_X''(0) = \frac{r(r-p+1)}{p^2}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{r(r-p+1)}{p^2} - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}$$

c) Outline of derivation: **(Extra Credit 2 pt)**

Consider the function $h(w) = (1-w)^{-r}$, the binomial (1 – w) with negative exponent -r

$$(1-w)^{-r} = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} = \sum_{0}^{\infty} \binom{r+k-1}{r-1} w^k$$

Let x = k + r in the summation, then k = x – r and

$$(1-w)^{-r} = \sum_{x=r}^{\infty} \binom{r+x-k-1}{r-1} w^{x-r} = \sum_{x=r}^{\infty} \binom{x-1}{r-1} w^{x-r}$$, as desired

4. X ~ Poisson (λ)

a) $M_X(t) = \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x e^{-\lambda e^t}}{x!} \cdot e^{-\lambda(1-e^t)}$, [we have defined a new Poisson (λe^t)] **(2 pt)**

$$= e^{-\lambda(1-e^t)} = \cdot e^{\lambda(e^t-1)}$$

b) $E(X) = M_X'(0) = [e^{\lambda(e^t-1)} \cdot \lambda e^t]|_{t=0} = \lambda$ **(2 pt)**

$$E(X^2) = M_X''(0) = [\lambda e^{t+\lambda(e^t-1)}(1+\lambda e^t)]|_{t=0} = \lambda + \lambda^2$$

$$Var(X) = E(X^2) - [E(X)]^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

5. Think in terms of a geometric distribution

a) Then the pmf of X is given by $p_X(x) = p(1-p)^{x-1} + (1-p)p^{x-1}$ **(1 pt)**

b) $M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx}\left[p(1-p)^{x-1} + (1-p)p^{x-1}\right]$ **(2 pt)**

$$= \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} + \sum_{x=1}^{\infty} e^{tx}(1-p)p^{x-1}$$

$$= \frac{pe^t}{1-(1-p)e^t} \sum_{x=1}^{\infty} (1-t)^{x-1} t + \frac{(1-p)e^t}{1-pe^t} \sum_{x=1}^{\infty} (1-v)^{x-1} v$$

$$= \frac{pe^t}{1-(1-p)e^t} + \frac{(1-p)e^t}{1-pe^t}$$

Note that this is the sum of two geometric MGF's and we know that $E(Y) = \frac{1}{p}$ for the regular geometric distribution. Therefore, the expected value is:

$E(X) = \frac{1}{p} + \frac{1}{1-p}$ {This result can also be obtained by finding $M'_X(0)$ }