Stat 331 HW7 Solutions (10' each)

3.22

Y~bin(12, 8/144), X~bin(12, 7/132), and independent
P[accepted]=P[Y=0]+P[Y=1]*P[X=0]=0.6885

3.25

$$
\begin{aligned}
P[X=x \mid X+Y=n] &= \frac{P[X=x, X+Y=n]}{P[X+Y=n]} \\
&= \frac{P[X=x, Y=n-x]}{P[X+Y=n]} \\
&= \frac{P[X=x]P[Y=n-x]}{P[X+Y=n]} \\
&= \frac{e^{-\lambda_1}\lambda_1^x}{x!} \times \frac{e^{-\lambda_2}\lambda_2^{n-x}}{(n-x)!} \times \frac{1}{\sum_{x=0}^{n} \frac{e^{-\lambda_1}\lambda_1^x}{x!} \times \frac{e^{-\lambda_2}\lambda_2^{n-x}}{(n-x)!}} \\
&\propto \frac{\lambda_1^x \lambda_2^{n-x}}{x!(n-x)!} \\
&\propto \binom{n}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-x}
\end{aligned}
$$

$$So,\ X \mid X+Y=n\ is\ bin(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$$

3.34

X~Unif[0, 30], Y~Unif[0, 45]
(X,Y)~Unif[0, 30; 0, 45], by independence

(a) $P[X>Y] = \iint_{\substack{X\in[0,30]\\ Y\in[0,45]\\ X>Y}} \frac{1}{30\times 45}dxdy = \frac{30\times 30}{30\times 45\times 2} = \frac{1}{3}$

(b) $P[|X-Y|>10] = \iint_{\substack{X\in[0,30]\\ Y\in[0,45]\\ |X-Y|>10}} \frac{1}{30\times 45}dxdy = \frac{(35+5)\times 30+20\times 20}{30\times 45\times 2} = \frac{16}{27} = 0.5926$

Note: This problem can also be done by total probability or conditional probability.

$$e.g.\ P[X>Y] = \int_0^{45} f[Y=y, X>y]dy = \int_0^{30} f(y)\int_y^{30} f(x)dxdy = \int_0^{30} \frac{1}{45}\times\frac{1}{30}\times(30-y)dy = \frac{1}{3}$$

3.90

Show $\rho(X,Y)=1 \Leftrightarrow Y=aX+b,\ a>0$

Since $Var\left[\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}\right]=2-2\rho$

"$\Leftarrow$"

$$Y=aX+b \Rightarrow Var\left[\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}\right]=\left(\frac{1}{\sigma_1}-\frac{a}{\sigma_2}\right)^2\sigma_1^2=0,\ \sin ce\ \sigma_2^2=a^2\sigma_1^2, a>0$$

$$\Rightarrow 2-2\rho=0$$

$$\Rightarrow \rho=1$$

"$\Rightarrow$"

$$\rho=1 \Rightarrow Var\left[\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}\right]=0$$

$$\Rightarrow \frac{X}{\sigma_1}-\frac{Y}{\sigma_2}=c$$

$$\Rightarrow Y=\frac{\sigma_2}{\sigma_1}X-c\sigma_2$$

$$\Rightarrow Y=aX+b,\ a=\frac{\sigma_2}{\sigma_1}>0,\ b=-c\sigma_2$$

$$Here,\ c=E[c]=E\left[\frac{X}{\sigma_1}-\frac{Y}{\sigma_2}\right]=\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2}$$