Stat 331 Solution 9 (40')

3.124 (10')

Claim: $\forall 1 \le i \le r, E[X_i] = np_i, \text{var}[X_i] = np_i(1-p_i)$

Note: another way is to consider $X_i \sim bin(n, p_i)$, as Page 263 says.

$\forall 1 \le j,k \le r, E[X_j X_k] = np_j p_k$

Proof:

$$since(X_1,...,X_r) \sim multinormial(n, p_1,..., p_r)$$

$$p[x_1,...,x_r] = \frac{n!}{x_1!...x_r!} p_1^{x_1}...p_r^{x_r}$$

$$\begin{aligned} So, E[X_i] &= \sum_{x_1,...,x_r=0}^{n} x_i \frac{n!}{x_1!...x_r!} p_1^{x_1}...p_r^{x_r} \\ &= \sum_{x_1,...,x_r} \frac{n!}{x_1!...(x_i-1)!...x_r!} p_1^{x_1}...p_r^{x_r} \\ &= np_i \sum_{x_1,...,x_r} \frac{(n-1)!}{x_1!...(x_i-1)!...x_r!} p_1^{x_1}...p_i^{x_i-1}...p_r^{x_r} \\ &= np_i \end{aligned}$$

$$\begin{aligned} Similarly, E[X_i(X_i-1)] &= \sum_{x_1,...,x_r} x_i(x_i-1)\frac{n!}{x_1!...x_r!} p_1^{x_1}...p_r^{x_r} \\ &= n(n-1)p_i^2 \end{aligned}$$

$$\begin{aligned} So, \text{var}[X_i] &= E[X_i(X_i-1)] + E[X_i] - (E[X_i])^2 \\ &= n(n-1)p_i^2 + np_i - (np_i)^2 \\ &= np_i(1-p_i) \end{aligned}$$

$$\begin{aligned} Similarly, E[X_j X_k] &= \sum_{x_1,...,x_r=0}^{n} x_j x_k \frac{n!}{x_1!...x_r!} p_1^{x_1}...p_r^{x_r} \\ &= n(n-1)p_j p_k \end{aligned}$$

$$\begin{aligned} So, \text{cov}[X_j, X_k] &= E[X_j X_k] - E[X_j]E[X_k] \\ &= n(n-1)p_j p_k - np_j np_k \\ &= -np_j p_k \end{aligned}$$

If using indicator (see Page 114), we need to consider

$$X_i \sim bin(n, p_i),\ X_j + X_k \sim bin(n, p_j + p_k),$$

Then:

$$E[X_i] = np_i$$

$$\text{var}[X_i] = np_i(1 - p_i)$$

$$\text{var}[X_j + X_k] = n(p_j + p_k)(1 - p_j - p_k)$$

$$So,\ \text{cov}[X_j, X_k] = \frac{\text{var}[X_j + X_k] - \text{var}[X_j] - \text{var}[X_k]}{2} = -np_j p_k$$

It is intuitively clear that the covariance is negative, because the increase of X_j will possibly cause the decrease of X_k since $X_1 + ... + X_r = n$. Thus X_j and X_k have negative relationship, i.e., the covariance is negative.

$$\rho = \frac{\text{cov}[X_j, X_k]}{\sqrt{\text{var}[X_j]\text{var}[X_k]}} = -\frac{np_j p_k}{\sqrt{np_j(1 - p_j)np_k(1 - p_k)}} = -\frac{1}{\sqrt{\left(\frac{1 - p_j}{p_j}\right)\left(\frac{1 - p_k}{p_k}\right)}}$$

3.155 (extra credit 2')

$X_S(t) \sim poi(2t), X_N(t) \sim poi(3t)$

(a)

$so, X_S(1') \sim poi(2), X_N(1') \sim poi(3)$

$since\ X_S\ and\ X_N\ are\ independent$

$(X_S + X_N)(1') \sim poi(5)$

$$thus, P[(X_S + X_N)(1') = 2] = \frac{e^{-5} 5^2}{2!}$$

(b)

$$P[X_S(1') = 1, X_N(1') = 1] = P[X_S(1') = 1] P[X_N(1') = 1]$$

$$= \frac{e^{-2} 2}{1!} \times \frac{e^{-3} 3}{1!} = 6e^{-5}$$

(c)

$$P[X_N(1') = 2 \mid (X_S + X_N)(1') = 2] = \frac{P[X_N(1') = 2, X_S(1') = 0]}{P[(X_S + X_N)(1') = 2]}$$

$$= \frac{\frac{e^{-3} 3^2}{2!} e^{-2}}{\frac{e^{-5} 5^2}{2!}}$$

$$= \frac{9}{25}$$

(d)

$(X_S + X_N)(10') \sim poi(50), (X_S + X_N)(5') \sim poi(25)$

$$P = \frac{[P[X_S + X_N)(5') = 2]]^2}{P[(X_S + X_N)(10') = 4]} = \frac{\left(\frac{e^{-25} 25^2}{2!}\right)^2}{\frac{e^{-50} 50^4}{4!}} = \frac{3}{8}$$

$$or, you\ can\ consider\ this\ as\ a\ bin(Y; 4, \frac{1}{2}), so\ P[Y = 2] = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

Additional problems:

1. (10')

$$a.\ M_{X_i}(t) = e^{\lambda(e^t-1)}$$

$$so, M_X(t) = [M_{X_i}(t)]^n = e^{n\lambda(e^t-1)}$$

$$thus, X \sim poi(n\lambda)$$

$$b.\ X_i \sim poi(\lambda/n) \Rightarrow X \sim poi(\lambda)$$

$$by\, CLT, \frac{X - n\times\frac{\lambda}{n}}{\sqrt{n}\times\sqrt{\frac{\lambda}{n}}} \xrightarrow{D} N(0,1),\ Note, \lambda\, should\, be\, large\, enough\, to\, make\, X_i\, meaningful$$

$$so, X \xrightarrow{D} N(\lambda,\lambda)$$

$$therefore, poi(\lambda)\, distribution\, can\, be\, approximated\, by\, N(\lambda,\lambda), when\, \lambda\, is\, large\, enough.$$

$$c.\ X \sim poi(24.9) \to N(24.9, 24.9)$$

$$P[X \geq 30] = P[\frac{X-24.9}{\sqrt{24.9}} \geq \frac{30-24.9}{\sqrt{24.9}}] \approx 1-\Phi(1) = 0.1587$$

2. (10')

a.

$$P[reject] = 1 - \frac{\frac{4}{3}\pi}{8} \approx 0.4764$$

$$E[N_{iter}] = \frac{1}{1-P[reject]} \approx 1.91$$

b.

$$\forall 0 \leq r \leq 1$$

$$P[\|R\cdot W\| \leq r] = P[R^3 \leq r^3]$$

$$also, P[\|R\cdot W\| \leq r] = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi} = r^3$$

$$so, P[R^3 \leq r^3] = r^3$$

$$generate\, U \sim unif[0,1], then\, R = \sqrt[3]{U}$$

$c.$

$Let\, Y_i = X_i^{1/3}, i = 1,2,3, then\, use\, the\, result\, of\, b\, to\, generate\,(y_1, y_2, y_3), hence\,(x_1, x_2, x_3)$

$d.$

$$for\, n\, odd, V = \frac{2^n \left(\frac{n-1}{2}\right)!}{n!} \pi^{\frac{n-1}{2}} r^n;\ for\, n\, even, V = \frac{1}{\left(\frac{n}{2}\right)!} \pi^{\frac{n}{2}} r^n$$

$generate\, U \sim unif[0,1], then\, R = \sqrt[n]{U}$

3. (5')

Proof:

$F_{F(X)}(x) = P[F(X) \le x] = P[X \le F^{-1}(x)] = F(F^{-1}(x)) = x$

$0 \le F(X) \le 1$

$so, F(X) \sim unif[0,1]$

4. (5')

This is a valid approach because it is equivalent to the rejection method.

$$X \sim truncated\, N(0,\sigma^2) \Rightarrow f(x) = C \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot 1_{[|x| \le \gamma]},\ C > 1, constant$$

$$Y \sim N(0,\sigma^2) \Rightarrow g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$\frac{f(x)}{g(x)} = C \cdot 1_{[|x| \le \gamma]}$$

$choose\ c = C,$

$$\frac{f(Y)}{cg(Y)} = 1_{[|Y| \le \gamma]}$$

$By\, rejection\, method,$

(1) $generate\, U \sim unif[0,1],\ Y \sim N(0,\sigma^2)$

(2) $Accept\, X = Y\, when\, U \le 1_{[|Y| \le \gamma]}, i.e., |Y| \le \gamma. Otherwise\, return\, to$ (1).