INDEPENDENCE

Independence of *two* events in $(\Omega, \mathcal{B}, P)$

$$P(AB) = P(A)P(B)$$

Independence of *a finite number of* events $A_1, \ldots, A_n$ in $(\Omega, , P)$

$$P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i),\ \forall I \subset \{1, \ldots, n\}$$

How many conditions are these?

Independence of finite number of classes of sets in $\mathcal{B}$

$\iff$ independence of all possible sets chosen "one from each class"

Independence of arbitrary set of classes of sets in $\mathcal{B}$

$\iff$ independence of all possible *finite subsets of classes*

Theorem 4.1.1. If

$$C_i, i = 1, \dots n,$$

are independent π-systems, then

$$\sigma(C_i),\ i = 1, \dots n,$$

are independent.

Proof (whiteboard).

Independent random variables

$$\forall t \in T, \quad X_t : (\Omega, \mathcal{B}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

$\{X_t,\ t \in T\}$ is an *independent family of rv's* iff $\{\sigma(X_t),\ t \in T\}$ are independent.

Examples

Theorem 4.2.1.

Family $\{X_t,\ t \in T\}$ is independent

iff all the finite dimnesional distribution functions factorize, i.e.,

$$\forall \text{ finite } J \subset T, \quad F_J(x_t,\ t \in J) := P[X_t \le x_t,\ t \in J] = \prod_{t \in J} P[X_t \le x_t]$$

Proof (whiteboard)

Examples of independence

Dyadic expansions (whiteboard and self-study)

Renyi Theorem

Assume $\{X_n,\ n \geq 1\}$ are iid with common *continuous* distribution function $F(x)$.

(a) The sequence of random variables (ranks) $\{R_n\ n \geq 1\}$ is independent and

$$P[R_n = k] = \frac{1}{n},\ k = 1, \ldots n.$$

(b) The sequence of events $\{A_n,\ n \geq 1\}$, where $A_n = [R_n = 1]$, is independent and

$$P[A_n] = \frac{1}{n}.$$

Proof

Due to continuity,

$$P[Ties] = 0$$

where

$$[Ties] := \bigcup_{i \neq j} [X_i = X_j].$$

(This requires a proof, see p. 97, whiteboard and self-study)

Therefore

$$R_n(\omega) = \sum_{j=1}^{n} 1_{[X_j \geq X_i]}(\omega)$$

Also, (b) $\Longrightarrow$ (a).

Since $\{X_n,\ n \geq 1\}$ are iid

$$P[X_{r_1} > X_{r_2} > \cdots > X_{r_n}] = \frac{1}{n!}$$

where

$$(r_1, r_2, \cdots, r_n) = permutation(1, 2, \ldots, n)$$

But we have an equivalence

$$[X_{r_1} > X_{r_2} > \cdots > X_{r_n}] \equiv [R_1 = r_1, R_2 = r_2, \ldots, R_n = r_n]$$

so

$$P[R_1 = r_1, R_2 = r_2, \ldots, R_n = r_n] = \frac{1}{n!}$$

Now

$$\begin{aligned} P[R_n &= r_n] = \sum_{r_1, r_2, \cdots, r_{n-1}} P[R_1 = r_1, R_2 = r_2, \ldots, R_n = r_n] \\ &= \sum_{r_1, r_2, \cdots, r_{n-1}} \frac{1}{n!} = (n-1)! \frac{1}{n!} = \frac{1}{n} \end{aligned}$$

hence

$$P[R_1 = r_1, R_2 = r_2, \ldots, R_n = r_n] = \prod_{i=1}^{n} P[R_i = r_i]$$

and $\{R_n \; n \geq 1\}$ independent

$\square$

Groupings

Independent family of σ-fields

$$\{\mathcal{B}_t,\ t \in T\}$$

Groups

$$\{T_s,\ s \in S\},\ T_{s_1} \cap T_{s_2} = \emptyset,\ s_1 \neq s_2$$

Grouped σ-fields

$$\mathcal{B}_{T_s} = \sigma\{\mathcal{B}_t, t \in T_s\}$$

Lemma

$$\{\mathcal{B}_{T_s},\ s \in S\}$$

is an independent family of σ-fields.

Proof. Self-study

Examples: (whiteboard and self-study)

Borel-Cantelli Lemma

If

$$\sum_n P(A_n) < \infty$$

then

$$P([A_n\ io]) = P(\lim \sup_{n \longrightarrow \infty} A_n) = 0$$

Proof

$$
\begin{aligned}
P([A_n\ io]) &= P(\bigcap_{n\geq 1}\bigcup_{j\geq n} A_j) = P(\lim_{n\longrightarrow\infty}\bigcup_{j\geq n} A_j) \\
&= \lim_{n\longrightarrow\infty} P(\bigcup_{j\geq n} A_j) \\
&\leq \lim_{n\longrightarrow\infty}\sup P(\bigcup_{j\geq n} A_j) \leq \lim_{n\longrightarrow\infty}\sup \sum_{j\geq n} P(A_j) \\
&= \lim_{n\longrightarrow\infty} \sum_{j\geq n} P(A_j) = 0
\end{aligned}
$$

□

Explanations and examples (whiteboard)

Borell Lemma

If $\{A_n,\ n \geq 1\}$ is an independent sequence of measurable sets (events), then

$$P([A_n\ io]) = \begin{cases} 0 & \iff \sum_n P(A_n) < \infty \\ 1 & \iff \sum_n P(A_n) = \infty \end{cases} .$$

Proof

We have to prove that $\sum_n P(A_n) = \infty \Longrightarrow P([A_n\ io]) = 1$ under independence.

$$
\begin{aligned}
P([A_n\ io]) &= P(\lim\sup_{n\longrightarrow\infty} A_n) = 1 - P(\lim\inf_{n\longrightarrow\infty} A_n^c) \\
&= 1 - \lim_{n\longrightarrow\infty} P(\cap_{k\geq n} A_k^c) = 1 - \lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty} P(\cap_{k=n}^m A_k^c) \\
&= 1 - \lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty}\prod_{k=n}^m [1 - P(A_k)] \\
&\geq 1 - \lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty}\prod_{k=n}^m \exp[-P(A_k)] \text{ (why?)} \\
&= 1 - \lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty}\exp[-\sum_{k=n}^m P(A_k)] \\
&\geq 1 - \lim_{n\longrightarrow\infty}\exp[-\sum_{k=n}^\infty P(A_k)] = 1 - \lim_{n\longrightarrow\infty} 0 = 1
\end{aligned}
$$

The proof is complete (??)

□

Kolmogorov 0-1 law

Tail events - events depending on the tail of a sequence of random variables, in a very abstract sense, i.e. not on any particular protion of the tail, but on the *"manner in which the tail behaves"*

$\{X_n,\ n \geq 1\}$, a sequence of rv's, define

$$\begin{aligned} \mathcal{F}_n &= \sigma(X_1, X_2, \ldots, X_n),\ \mathcal{F}_n \uparrow \\ \mathcal{F}'_n &= \sigma(X_{n+1}, X_{n+2}, \ldots),\ \mathcal{F}'_n \downarrow \end{aligned}$$

$$\mathcal{F}_\infty = \sigma(X_1, X_2, \ldots, X_n, \ldots) = \sigma(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots)$$

and finally, the tail σ-field

$$\mathcal{T} = \lim_{n \longrightarrow \infty} \downarrow \sigma(X_{n+1}, X_{n+2}, \ldots) = \bigcap_n \mathcal{F}'_n$$

Examples (details, whiteboard)

$$\left\{\omega : \sum_n X_n(\omega) \text{ converges}\right\} \in \mathcal{T}$$

$$\left\{\omega : \lim_n X_n(\omega) \text{ converges}\right\} = \left\{\omega : \liminf_n X_n(\omega) = \limsup_n X_n(\omega)\right\} \in \mathcal{T}$$

Theorem 4.5.3. Suppose we have an iid sequence of rv's

$$\{X_n,\ n \geq 1\}$$

Then

$$\Lambda \in \mathcal{T} \implies P(\Lambda) = \begin{cases} \quad 0 \\ \text{or} \\ \quad 1 \end{cases}$$

or in other words

$$P(\Lambda \cap \Lambda) = P(\Lambda) = P(\Lambda)^2$$

ie. Λ is independent of itself.

σ-algebra composed of sets Λ

such that $P(\Lambda) = \begin{cases} 0 \\ \text{or} \\ 1 \end{cases}$ is called *almost trivial*

Proof

We have

$$\Lambda \in \mathcal{T} \subset \mathcal{F}'_n \subset \mathcal{F}_\infty$$

hence

$$\mathcal{F}'_n \perp \mathcal{F}_n \Longrightarrow \Lambda \perp \mathcal{F}_n \Longrightarrow \Lambda \perp \bigcup_n \mathcal{F}_n$$

Set

$$\mathcal{C}_1 = \{\Lambda\} \perp \bigcup_n \mathcal{F}_n = \mathcal{C}_2$$

therefore

$$\Lambda \in \{\emptyset, \Omega, \Lambda, \Lambda^c\} = \sigma\{\Lambda\} \perp \sigma\{\bigcup_n \mathcal{F}_n\} = \mathcal{F}_\infty \ni \Lambda$$

Consequences

Lemma 4.5.1.

Let $\mathcal{G}$ be almost trivial σ-field and let X be a rv measurable wrt $\mathcal{G}$. Then

$$P[X = c] = 1$$

for some constant c.

Proof, whiteboard

Corollary 4.5.1, whiteboard