Some overdue derivations

Define the left-continuous (how do we know?) inverse of F

$$F^{\longleftarrow}(y) = \inf\{s : F(s) \geq y\},\ 0 < y \leq 1$$

$$A(y) = \{s : F(s) \geq y\}$$

Properties of $A(y)$*:*

(a) $A(y)$ is closed

Indeed

$$A(y) \ni s_n \downarrow s \implies F(s_n) \downarrow F(s) \geq y \implies s \in A(y)$$

$$A(y) \ni s_n \uparrow s \implies y \leq F(s_n) \uparrow F(s-0) \leq F(s) \implies s \in A(y)$$

□

(b) Therefore,

$$\inf A(y) \in A(y) \Longleftrightarrow F(F^{\longleftarrow}(y)) \geq y$$

The part on the left follows by closedness.

But this means that the defining property $F(s) \geq y$ holds for the $s = \inf A(y) = F^{\longleftarrow}(y)$

□

(c) Furthermore

$$[F^{\leftarrow}(y) > t \iff y > F(t)] \iff [F^{\leftarrow}(y) \leq t \iff y \leq F(t)]$$

Indeed,

$$t < F^{\leftarrow}(y) \iff t \notin A(y) \iff F(t) < y$$

$$t \geq F^{\leftarrow}(y) \iff t \in A(y) \iff F(t) \geq y$$

□

Now we can prove that $F^{\longleftarrow}(y)$ *is left continuous*

$$y_n \uparrow y \implies F^{\longleftarrow}(y_n) \le F^{\longleftarrow}(y_{n+1}) \le F^{\longleftarrow}(y)$$

Therefore

$$\phi = \lim_{n \uparrow \infty} F^{\longleftarrow}(y_n) \le F^{\longleftarrow}(y)$$

How to exclude $\phi < F^{\longleftarrow}(y)$?

$$\begin{aligned} \forall n \quad &: \quad F^{\longleftarrow}(y_n) \le \phi \overset{(c)}{\Longrightarrow} y_n \le F(\phi) \\ \Longrightarrow y \quad &\le \quad F(\phi) \overset{(c)}{\Longrightarrow} F^{\longleftarrow}(y) \le \phi \end{aligned}$$

□

RANDOM VARIABLES, ELEMENTS AND MEASURABLE MAPS

Intuitive introduction (blackboard)

- Reproducible repeat experiments
- How to measure outcomes of experiments?
- Intuitive notion of random variable

- Random variable as a function
- Mathematical conditions to be satisfied by random variables: measurability
- Induced probability measures = distributions of rv's
- Generalizations: Stochastic processes

Inverse Maps

$$X : \Omega \longrightarrow \Omega'$$

$$X^{-1} : 2^{\Omega'} \longrightarrow 2^{\Omega} : X^{-1}(A') = \{\omega \in \Omega : X(\omega) \in A'\}$$

(drawing ...)

Properties (prove ...):

(i)

$$X^{-1}(\emptyset) = \emptyset;\ X^{-1}(\Omega') = \Omega$$

(ii)

$$X^{-1}(A'^c) = X^{-1}(A')^c$$

(iii)

$$X^{-1}(\bigcup_t A'_t) = \bigcup_t X^{-1}(A'_t)$$

Indeed

$$\begin{aligned} X^{-1}\Big(\bigcup_t A'_t\Big) &= \{\omega \in \Omega : X(\omega) \in \bigcup_t A'_t\} \\ &= \{\omega \in \Omega : [\ \exists t : X(\omega) \in A'_t\ ]\} \\ &= \{\omega \in \Omega : [\ \exists t : \omega \in X^{-1}(A'_t)\ ]\} \\ &= \bigcup_t X^{-1}(A'_t) \end{aligned}$$

(iv)

$$X^{-1}\Big(\bigcap_t A'_t\Big) = \bigcap_t X^{-1}(A'_t)$$

Notation

If $\mathcal{C}' \subset 2^{\Omega'}$

$$X^{-1}(\mathcal{C}') := \{X^{-1}(C') : C' \in \mathcal{C}'\}$$

Proposition 3.1.1. If $\mathcal{B}'$ is a σ-field of subsets of Ω', then $X^{-1}(\mathcal{B}')$ is a σ-field of subsets of Ω.

Proof

(i)

$$\Omega' \in \mathcal{B}' \implies \Omega = X^{-1}(\Omega') \in X^{-1}(\mathcal{B}')$$

(ii)

$$A' \in \mathcal{B}' \implies X^{-1}(A') \in X^{-1}(\mathcal{B}')$$

also

$$A' \in \mathcal{B}' \implies A'^c \in \mathcal{B}' \implies X^{-1}(A'^c) = X^{-1}(A')^c \in X^{-1}(\mathcal{B}')$$

therefore

$$X^{-1}(A') \in X^{-1}(\mathcal{B}') \implies X^{-1}(A')^c \in X^{-1}(\mathcal{B}')$$

(iii)

$$X^{-1}(A'_n) \in X^{-1}(\mathcal{B}') \implies \bigcup_n X^{-1}(A'_n) = X^{-1}(\bigcup_n A'_n) \in X^{-1}(\mathcal{B}')$$

□

Proposition 3.1.2. If $\mathcal{C}' \in 2^{\Omega'}$,then

$$X^{-1}(\sigma(\mathcal{C}')) = \sigma(X^{-1}(\mathcal{C}'))$$

Proof self-study

Measurable maps, random elements, induced probability measures

Measurable space: $(\Omega, \mathcal{B})$

Measurable map: Given $(\Omega, \mathcal{B})$ and $(\Omega', \mathcal{B}')$, a map

$$X : \Omega \longrightarrow \Omega'$$

is called measurable if

$$X^{-1}(\mathcal{B}') = \mathcal{B}$$

X is also called a random element of Ω':

$$X \in \mathcal{B}/\mathcal{B}' \iff X : (\Omega, \mathcal{B}) \longrightarrow (\Omega', \mathcal{B}')$$

This latter is by abuse of notation (why?)

Random variable is a special case of random element when $(\Omega', \mathcal{B}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Random vector is a special case of random element when $(\Omega', \mathcal{B}') = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

Induced probabilities

Let $(\Omega, \mathcal{B}, P)$ be a probability space

and $X : (\Omega, \mathcal{B}) \longrightarrow (\Omega', \mathcal{B}')$ a measurable map

Define

$$[X \in A'] := X^{-1}(A') = \{\omega : X(\omega) \in A'\}$$

and a set function $P \circ X^{-1}$ on $\mathcal{B}'$ by

$$P \circ X^{-1}(A') = \text{``}(P \circ X^{-1})(A')'' = P(X^{-1}(A'))$$

Is $P \circ X^{-1}$ a (*induced*) probability measure on $\mathcal{B}'$?

Yes!

Verify ...(!)

Notation and jargon

$$P \circ X^{-1}(A') = P[X \in A']$$

If X is a rv (random variable), then $P \circ X^{-1}$ is a measure induced on $\mathbb{R}$ by the distribution function

$$P \circ X^{-1}(-\infty, x] = P[X \leq x] = F_X(x)$$

Induced distribution is a condensed summary of the probability measure on $(\Omega, \mathcal{B})$, which is sufficient to describe all events (measurable sets) induced by rv X.

Example: Counts of nucleotides in a random DNA sequence

Consider a DNA sequence of length N.

Each site can be independently occupied by A, C, G, or T equally likely.

$$\Omega = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N),\ \omega_i \in \{A, C, G, T\}\}$$

$$\mathcal{A} = \sigma(\{\omega\}) = 2^{\Omega}$$

The set function

$$P(A) = \#\{A\} \cdot 4^{-N},\ A \in \mathcal{A}$$

satisfies axioms of probability measure (this is an example of combinatorial probability).

Define

$$X : \Omega \to \Omega' := \{0, 1, 2, \ldots, N\}$$

$$X(\omega) = \#\{G \text{ in } \omega\}$$

$$X^{-1}(\{0\}) = \{(\tilde{\ }G, \tilde{\ }G, \ldots, \tilde{\ }G)\}$$

etc.

$$
\begin{aligned}
P'(\{i\}) &= P \circ X^{-1}(\{i\}) \\
&= P(X^{-1}(\{i\})) \\
&= \begin{cases}
(3/4)^N & i = 0 \\
(1/4)(3/4)^{N-1} & i = 1 \\
\cdots & \\
\binom{N}{i}(1/4)^i(3/4)^{N-i} & i \\
\cdots & \\
(1/4)^N & i = N
\end{cases}
\end{aligned}
$$

Test for measurability

Proposition 3.2.1. Suppose

$$X : \Omega \to \Omega'$$

where $(\Omega, \mathcal{B})$ and $(\Omega', \mathcal{B}')$ are measurable spaces. Suppose

$$\mathcal{B}' = \sigma(\mathcal{C}')$$

Then X is measurable iff

$$X^{-1}(\mathcal{C}') \subset \mathcal{B}$$

i.e. we do not have to check

$$X^{-1}(\sigma(\mathcal{C}')) \subset \mathcal{B}$$

Proof by counterimage properties (self-study)

Corollary 3.2.1. The real-valued function

$$X : \Omega \to \mathbb{R}$$

is a random variable iff

$$X^{-1}((-\infty, \lambda]) = [X \leq \lambda] \in \mathcal{B},\ \forall \lambda \in \mathbb{R}$$

Topics for whiteboard explanation

- Composition of measurable maps
- Random elements of metric spaces
- Measurability and continuity
- Maesurability and limits
- σ-fields generated by maps

σ-fields generated by maps

Consider rv $X : (\Omega, \mathcal{B}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

The σ-field generated by X

$$\sigma(X) = X^{-1}[\mathcal{B}(\mathbb{R})]$$

or

$$\sigma(X) = \{[X \in A],\ A \in \mathcal{B}(\mathbb{R})\} = \textbf{(!)}\ \sigma\{[X \in A],\ A \in \mathcal{B}(\mathbb{R})\}$$

More general construction, if $X : (\Omega, \mathcal{B}) \longrightarrow (\Omega', \mathcal{B}')$

$$\sigma(X) = X^{-1}(\mathcal{B}')$$

If $\mathcal{F} \subset \mathcal{B}$, is a sub-$\sigma$-field of $\mathcal{B}$ and

$$\sigma(X) \subset \mathcal{F} \subset \mathcal{B}$$

then we say X is measurable wrt $\mathcal{F}$.

σ-fields generated by an indexed set of maps (ie. by almost a process)

Take $t \in T$. Then

$$X_t : (\Omega, \mathcal{B}) \longrightarrow (\Omega', \mathcal{B}')$$

is a family of maps (almost random elements), and

$$\sigma(X_t,\ t \in T) = \vee_{t\in T}\sigma(X_t) = \sigma\left[\cup_{t\in T}\sigma(X_t)\right]$$

is the smallest σ-field containing all $\sigma(X_t)$.

Examples:

$X = c$, $X = 1_A$ (whiteboard)

Further examples:

Simple functions, σ-fields generated by sequences of rv's

Proposition 3.3.1. X is a rv and $\mathcal{C} \in 2^{\mathbb{R}}$ st

$$\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$$

Then

$$\sigma(X) = \sigma\{[X \in B],\ B \in \mathcal{C}\}$$

Proof

$$\begin{aligned}
\sigma\{[X \in B],\ B \in \mathcal{C}\} &= \sigma\{X^{-1}(B),\ B \in \mathcal{C}\} \\
&= \sigma\{X^{-1}(\mathcal{C})\} \\
&= X^{-1}\{\sigma(\mathcal{C})\} \\
&= X^{-1}\{\mathcal{B}(\mathbb{R})\} \\
&= \sigma(X)
\end{aligned}$$

$\square$