SET THEORY CONCEPTS

Cardinality (power) of a set = "number of elements in a set"

$\#\{A\} = \#\{B\}$ iff elements of A and B can be paired i.e., there exists a function f that is 1-1 and "on" such that for each $a \in A$ and $b \in B$

$$a = f(b),\ b = f^{-1}(a)$$

Countable set has cardinality the same as the set N of natural numbers. This means that the elements of a countable set can be indexed by natural numbers

$$A = \{a_n,\ n = 1, 2, \ldots\}.$$

Examples of countable sets: Z (integers), Q (rational numbers)

Cardinal numbers describe cardinalities of sets

$$\begin{aligned} \aleph_0 &= \#\{N\} \\ \mathfrak{C} &= \#\{[0,1]\} \\ \mathfrak{C} &\geq \aleph_0 \end{aligned}$$

Power set of A is the set of all subsets of A

$$\mathcal{P}(A) = 2^A = \{X : X \subseteq A\}$$

We have (Theorem) $\#\{2^A\} > \#\{A\}$. Indeed, suppose $\#\{2^A\} = \#\{A\}$, i.e., elements of 2^A and A can be paired. But this means that if $f : 2^A \rightarrow A$ then for any $x \in A$, $f(\{x\}) = y$, so that all $y \in A$ are exhausted. But there are elements in 2^A, which are different from singletons $\{x\}$ (contradiction).

We have $\#\{2^N\} = \#\{[0,1]\}$. Indeed, all subsets of N can be coded by sequences of 0 and 1. For example, even numbers are coded as $01010101\ldots$. These codings are binary expansions of numbers from $[0,1]$, so they provide f.

Consequence (Corollary) is that

$$\mathfrak{C} > \aleph_0$$

Continuum hypothesis, there is no cardinal number between $\mathfrak{C}$ and $\aleph_0$.

Limit operations on sets

Consider a family of sets indexed by natural numbers $\{A_n, n = 1, 2, \ldots\}$.

$$
\begin{aligned}
H_k &= \cup_{n \geq k} A_n, \Rightarrow H_{k+1} \subseteq H_k, \text{ descending family} \\
G_k &= \cap_{n \geq k} A_n, \Rightarrow G_{k+1} \supseteq G_k, \text{ ascending family} \\
G_k &\subseteq H_k
\end{aligned}
$$

We write

$$
\begin{aligned}
\limsup A_n &= \lim_{k \to \infty} H_k = \cap_{k \geq 1} \cup_{n \geq k} A_n \\
\liminf A_n &= \lim_{k \to \infty} G_k = \cup_{k \geq 1} \cap_{n \geq k} A_n
\end{aligned}
$$

Obviously (why?)

$$\liminf A_n \subseteq \limsup A_n$$

If $\liminf A_n = \limsup A_n$, then they define $\lim A_n$ (remember A_n are not necessarily ascending/descending).

Interpretation

$$\begin{aligned}\limsup A_n &= \{\omega : \forall k, \exists (n \geq k), \omega \in A_n\} \\ &= \{\omega \text{ in infinitely many } A_n\} \\ &= \{A_n, \text{ i.o.}\}\end{aligned}$$

$$\begin{aligned}\liminf A_n &= \{\omega : \exists k, \forall (n \geq k), \omega \in A_n\} \\ &= \{\omega \text{ in all, except for final number, of } A_n\}\end{aligned}$$

Attention, $k = k(\omega), n = n(\omega)$.

Indicator functions of sets

$$1_A(\omega) = \begin{cases} 1; & \omega \in A \\ 0; & \omega \notin A \end{cases}$$

Calculus of indicator functions

$$1_{A\cup B}(\omega) = 1_A(\omega) \vee 1_B(\omega) = \max\{1_A(\omega), 1_B(\omega)\}$$

$$1_{A\cap B}(\omega) = 1_A(\omega) \wedge 1_B(\omega) = \min\{1_A(\omega), 1_B(\omega)\}$$

$$1_{A^c}(\omega) = 1 - 1_A(\omega)$$

$$\begin{aligned} 1_{\limsup A_n}(\omega) &= 1_{\cap_{k\geq 1}\cup_{n\geq k} A_n}(\omega) \\ &= \wedge_{k\geq 1} \vee_{n\geq k} 1_{A_n}(\omega) \\ &= \limsup 1_{A_n}(\omega) \end{aligned}$$

and the dual form for $1_{\liminf A_n}(\omega)$ ...