Stat581 HW1 Solutions

1. (3')

(a) 15.

(b) Proof: Since the order of partial derivatives doesn't matter, this problem is equivalent to assign r identical balls to n identical boxes. Note that there are n-1 barriers, so the total number of ways is $\binom{n+r-1}{r}$.

2. (3')

By inclusive-exclusive theorem,

$$
\begin{aligned}
P(\text{at least 1 call per day}) &= 1 - P(\text{exist some days with no call}) \\
&= 1 - \binom{7}{1}\frac{6^{12}}{7^{12}} + \binom{7}{2}\frac{5^{12}}{7^{12}} - \ldots + \binom{7}{6}\frac{1^{12}}{7^{12}} \\
&= 0.228452
\end{aligned}
$$

This can also be done in another way. Consider $X_1 \sim bin(12, \frac{1}{7})$, then

$P(X_1 = 0)$ represents the probability of no call received in a single day.

Consider $X_2 \sim bin(12, \frac{1}{6})$, then $P(X_1 = 0)P(X_2 = 0)$ represents the probability of no call received in two days …

So, $P = 1 - \binom{7}{1}P(X_1 = 0) + \binom{7}{2}P(X_1 = 0)P(X_2 = 0) - \ldots$

Direct approach (by Pawell):

A sample point specifies on which day (1 through 7) each of the 12 calls happens. Thus there are 7^{12} equally likely sample points. There are several different ways that the calls might be assigned so that there is at least one call each day. There might be 6 calls one day and 1 call each of the other days. Denote this by 6111111. The number of sample points with this pattern is $7\binom{12}{6}6!$. There are 7 ways to specify the day with 6 calls. There are $\binom{12}{6}$ to specify which of the 12 calls are on this day. And there are 6! ways of assigning the remaining 6 calls to the remaining 6 days. We will now count another pattern. There might be 4 calls on one day, 2 calls on each of two days, and 1 call on each of the remaining four days. Denote this by 4221111. The number of sample points with this pattern is $7\binom{12}{4}\binom{6}{2}\binom{8}{2}\binom{6}{2}4!$. (7 ways to pick day with 4 calls, $\binom{12}{4}$ to pick the calls for that day, $\binom{6}{2}$ to pick two days with two calls, $\binom{8}{2}$ ways to pick two calls for lowered numbered day, $\binom{6}{2}$ ways to pick the two calls for higher numbered day, 4! ways to order remaining 4 calls.) Here is a list of all the possibilities and the counts of the sample points for each one.

pattern	number of sample points	
6111111	$7\binom{12}{6}6! =$	4,656,960
5211111	$7\binom{12}{5}6\binom{7}{2}5! =$	83,825,280
4221111	$7\binom{12}{4}\binom{6}{2}\binom{8}{2}\binom{6}{2}4! =$	523,908,000
4311111	$7\binom{12}{4}6\binom{8}{3}5! =$	139,708,800
3321111	$\binom{7}{2}\binom{12}{3}\binom{9}{3}5\binom{6}{2}4! =$	698,544,000
3222111	$7\binom{12}{3}\binom{6}{3}\binom{9}{3}\binom{7}{2}\binom{5}{2}3! =$	1,397,088,000
2222211	$\binom{7}{5}\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}2! =$	314,344,800
		3,162,075,840

The probability is the total number of sample points divided by 7^{12}, which is $\frac{3{,}162{,}075{,}840}{7^{12}} \approx .2285$.

3. (3')

$P(\text{no matching pair}) = \dfrac{\binom{n}{2r} 2^{2r}}{\binom{2n}{2r}}$, since we should choose 2r pairs from n pairs first to guarantee there is no matching pair, and then choose 1 from each of the 2r pairs.

4. (1')

$$P(\text{both are boys} \mid \text{at least one is boy}) = \frac{P(\text{both are boys})}{P(\text{at least one is boy})} = \frac{1/4}{1-1/4} = \frac{1}{3}.$$

5. (2')

$X \sim NegBin(1, \frac{1}{6}), P(X \geq 5) = 1 - P(X < 4) = 1 - 0.5981 = 0.4019$.

Or, let B_i ={cast i times, the last one is 6},

$P(\text{cast more than 5 times}) = 1 - P(B_i \leq 5) = 1 - (\frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} + \ldots + (\frac{5}{6})^4 \cdot \frac{1}{6})$

Or directly, $P(\text{cast more than 5 times}) = (\frac{5}{6})^5$.

6. (2')

$X \sim Bin(10, 0.2)$,

$P(X \geq 2) = 1 - P(X < 2) = 0.6242$,

$P(X \geq 2 \mid X \geq 1) = \dfrac{P(X \geq 2)}{P(X \geq 1)} = 0.6993$.

7. (4')

Show (1) right continuous, (2) monotone non-decreasing, (3) $F(-\infty) = 0, F(\infty) = 1$.

8. (4')

Use change of integrals.

(a) Proof:

$$\int_0^\infty [1 - F_X(x)]dx = \int_0^\infty \int_x^\infty f(t)dtdx = \int_0^\infty \int_0^t f(t)dxdt = \int_0^\infty tf(t)dt = E(X)$$

(b) Proof:

$$\sum_{k=0}^\infty [1 - F_X(k)] = \sum_{k=0}^\infty P(X > k) = \sum_{k=0}^\infty \sum_{t=k+1}^\infty P(X = t) = \sum_{t=1}^\infty \sum_{k=0}^{t-1} P(X = t) = \sum_{t=1}^\infty tP(X = t) = E(X)$$

9. (3')

Easy to show the hint, $\forall \omega, (X + Y)(\omega) = (X \vee Y)(\omega) + (X \wedge Y)(\omega)$.

$X + Y = (X \vee Y) + (X \wedge Y) \Rightarrow E(X + Y) = E(X) + E(Y) = E(X \vee Y) + E(X \wedge Y)$.

10. (2’)

$$E(X)=\int_0^1 ax^a dx=\frac{a}{a+1},$$

$$E(X^2)=\int_0^1 ax^{a+1} dx=\frac{a}{a+2},$$

$$Var(X)=E(X^2)-E(X)=\frac{a}{(a+1)^2(a+2)}$$

11. (3’)

(a) Show $P(X=i,Y=j)\neq P(X=i)P(Y=j), \text{for some } i=1,2,3,\ j=2,3,4$

(b)

	1	2	3
2	1/12	1/6	1/12
3	1/12	1/6	1/12
4	1/12	1/6	1/12

12. (3’)

Proof:

Let $U=\frac{X}{X+Y}, V=X$, calculate joint distribution after transformation and then the marginal.

13. (3’)

Proof:

$$\begin{aligned}P(X=x\mid X+Y=n)&=\frac{P(X=x,X+Y=n)}{P(X+Y=n)}\\&=\frac{P(X=x,Y=n-x)}{P(X+Y=n)}\\&=\frac{P(X=x)P(Y=n-x)}{P(X+Y=n)}\\&=\binom{n}{x}\left(\frac{\theta}{\theta+\lambda}\right)^x\left(\frac{\lambda}{\theta+\lambda}\right)^{n-x}, 0\le x\le n\end{aligned}$$