Stat581 HW2 Solutions

4. (4') Suppose $A_n = \{\frac{m}{n} : m \in N\}, n \in N$, where N are non-negative integers. What is $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$?

(1) show $\inf_{k\geq n} A_k = N$,

$$\left.\begin{array}{l} \inf_{k\geq n} A_k = \bigcap_{k=n}^{\infty} A_k \subset N \text{ (not obvious, but right)} \\ N \subset A_k, \forall k \geq n \Rightarrow N \subset \bigcap_{k=n}^{\infty} A_k = \inf_{k\geq n} A_k \end{array}\right\} \Rightarrow \inf_{k\geq n} A_k = N,$$

hence, $\liminf_{n\to\infty} A_n = \lim_{n\to\infty}(\inf_{k\geq n} A_k) = N.$

(2) show $\sup_{k\geq n} A_k = Q^+$,

$$\left.\begin{array}{l} \sup_{k\geq n} A_k = \bigcup_{k=n}^{\infty} A_k = \{\frac{m}{g}, g \geq n, m \in N\} \subset Q^+ \\ \forall q \in Q^+, q \in \{\frac{m}{g}, m \in N\} = \{\frac{m}{g}, g \geq n, m \in N\} = \sup_{k\geq n} A_k \end{array}\right\} \Rightarrow \sup_{k\geq n} A_k = Q^+,$$

hence, $\limsup_{n\to\infty} A_n = \lim_{n\to\infty}(\sup_{k\geq n} A_k) = Q^+.$

5. (4') Let f_n, f be real functions on Ω . Show

$$\{\omega : f_n(\omega) \not\to f(\omega)\} = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| \geq \frac{1}{k}\} .$$

Proof:

By De Morgan's Law, it's equivalent to show

$$\{\omega : f_n(\omega) \to f(\omega)\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| < \frac{1}{k}\}.$$

Now, using $\varepsilon - \delta$ language,

$$\forall \omega \in \{\omega : f_n(\omega) \to f(\omega)\}, \forall k \geq 1, \exists N, \forall n \geq N, |f_n(\omega) - f(\omega)| < \frac{1}{k},$$

so, $\forall k \geq 1, \exists N, \omega \in \bigcap_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| < \frac{1}{k}\}$,

hence, $\forall k \geq 1, \omega \in \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| < \frac{1}{k}\}$,

therefore, $\omega \in \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| < \frac{1}{k}\}$.

Similarly, we can show $\forall \omega \in \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| < \frac{1}{k}\}, f_n(\omega) \to f(\omega)$.

6. (4') Suppose $a_n > 0, b_n > 1$ and $\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 1$.

Define $A_n = \{x : a_n \le x < b_n\}$. Find $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

$$\limsup_{n\to\infty} A_n = \bigcap_{n\ge1}\bigcup_{k\ge n}[a_k, b_k) = \bigcap_{n\ge1}(\inf_{k\ge n} a_k, \sup_{k\ge n} b_k) = (\sup_{n\ge1}\uparrow(\inf_{k\ge n} a_k), \inf_{n\ge1}\downarrow(\sup_{k\ge n} b_k)]$$

$$= (\lim_{n\ge1}(\inf_{k\ge n} a_k), \lim_{n\ge1}(\sup_{k\ge n} b_k)] = (\lim_{n\ge1} a_n, \lim_{n\ge1} b_n] = (0,1],$$

note that $0 \notin \limsup_{n\to\infty} A_n$, and $1 \in \limsup_{n\to\infty} A_n$ since 1 appears infinitely often while 0 not.

you may also want to guess $(0,1]$ to be the answer then verify every element is i.o.

similarly, $\liminf_{n\to\infty} A_n = (0,1]$, note here $1 \in \liminf_{n\to\infty} A_n$ also, since $1 \in A_n$ except a finite number.

9. (2') Check that $A\Delta B = A^c \Delta B^c$.

$$A\Delta B = (A\bigcup B)\setminus(A\bigcap B) = (A\bigcup B)\bigcap(A\bigcap B)^c$$

$$\overset{\text{De Morgan}}{=} (A^c\bigcap B^c)^c\bigcap(A^c\bigcup B^c) = (A^c\bigcup B^c)\setminus(A^c\bigcap B^c) = A^c\Delta B^c.$$

And many other methods may apply also.

12. (2') Let $\Omega = \{1,2,3,4,5,6\}$ and let $\mathcal{C} = \{\{2,4\},\{6\}\}$. What is the field generated by $\mathcal{C}$ and what is the $\sigma-$field?

Since $\mathcal{C}$ partitions Ω into 3 parts, there are 8 sets in $\sigma(\mathcal{C})$, and since Ω is finite, the field generated by $\mathcal{C}$ is equivalent to $\sigma(\mathcal{C})$.

$\sigma(\mathcal{C}) = \{\varnothing, \Omega, \{2,4\}, \{6\}, \{2,4,6\}, \{1,3,5,6\}, \{1,2,3,4,5\}, \{1,3,5\}\}$.

40. (4') Show that $\mathcal{B}((0,1])$ is generated by the following countable collection: For an integer r, $\{[kr^{-n}, (k+1)r^{-n}), 0 \le k < r^n, n = 1,2,\ldots\}$.

Proof: (here, assume k is also integer)

(1) show $\mathcal{B}((0,1]) = \sigma(\mathcal{C}(0,1]) = \sigma([a,b), 0 \le a < b \le 1)$, following the steps in book P17.

(2) show $\sigma([kr^{-n}, (k+1)r^{-n})) = \sigma([a,b), 0 \le a < b \le 1)$

(you may want to show (a) $\{[kr^{-n}, (k+1)r^{-n})\} \subset \sigma([a,b))$ and

(b) $\sigma([a,b))$ is contained by any other $\sigma-$field that contains $\{[kr^{-n}, (k+1)r^{-n})\}$,

but this is equivalent to show $\sigma([a,b)) = \sigma([kr^{-n}, (k+1)r^{-n}))$ directly)

(i) $\forall k, [kr^{-n}, (k+1)r^{-n}) \in \sigma([a,b), 0 \le a < b \le 1) \Rightarrow \sigma([kr^{-n}, (k+1)r^{-n})) \subset \sigma([a,b), 0 \le a < b \le 1)$

(ii) $\forall 0 \le a < b \le 1, \exists n_i$ and $0 \le k_i < r^{-n_i}$, s.t., $[a,b) = \bigcup_{i=1}^{\infty}[k_i r^{-n_i}, (k_i+1)r^{-n_i})$,

note this is not that obvious if a or b is irrational, but it's true.

therefore $[a,b) \in \sigma([kr^{-n}, (k+1)r^{-n}))$ and hence $\sigma([a,b), 0 \le a < b \le 1) \subset \sigma([kr^{-n}, (k+1)r^{-n}))$

Extra: (4') prove inclusion-exclusion formula (2.2) by induction, starting from n=2.

Proof:

(1) For $n = 2$, easy to show by (2.1);

(2) assume (2.2) holds for n, that is

$$P(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} P(A_j) - \sum_{1\le i<j\le n} P(A_i A_j) + \ldots + (-1)^{n+1} P(A_1 \cdots A_n) \qquad (*),$$

then for $n+1$,

$$P(\bigcup_{j=1}^{n+1} A_j) = P[(\bigcup_{j=1}^{n} A_j) \bigcup A_{n+1}] \overset{(*)}{=} P(\bigcup_{j=1}^{n} A_j) + P(A_{n+1}) - P[(\bigcup_{j=1}^{n} A_j) \bigcap A_{n+1}]$$

$$= \sum_{j=1}^{n} P(A_j) - \sum_{1\le i<j\le n} P(A_i A_j) + \ldots + (-1)^{n+1} P(A_1 \cdots A_n) + P(A_{n+1}) - P[(\bigcup_{j=1}^{n} A_j) \bigcap A_{n+1}]$$

$$= \sum_{j=1}^{n+1} P(A_j) - \sum_{1\le i<j\le n} P(A_i A_j) + \ldots + (-1)^{n+1} P(A_1 \cdots A_n) - P[\bigcup_{j=1}^{n} (A_j \bigcap A_{n+1})]$$

$$\overset{(*)}{=} \sum_{j=1}^{n+1} P(A_j) - \sum_{1\le i<j\le n} P(A_i A_j) + \ldots + (-1)^{n+1} P(A_1 \cdots A_n) -$$

$$[\sum_{j=1}^{n} P(A_j \bigcap A_{n+1}) - \sum_{1\le i<j\le n} P(A_i A_j \bigcap A_{n+1}) + \ldots + (-1)^{n+1} P(A_1 \cdots A_n \bigcap A_{n+1})]$$

$$= \sum_{j=1}^{n+1} P(A_j) - \sum_{1\le i<j\le n+1} P(A_i A_j) + \ldots + (-1)^{n+2} P(A_1 \cdots A_{n+1})$$