Stat581 HW4 Solutions

2.(3') Let $(\Omega, \mathcal{B}, P) = ((0,1], \mathcal{B}(0,1], \lambda)$ where λ is a Lebesgue measure. Define

$$\begin{aligned} X_1(\omega) &= 0, \forall \omega \in \Omega, \\ X_2(\omega) &= 1_{1/2}(\omega), \\ X_3(\omega) &= 1_Q(\omega) \end{aligned}$$

where $\mathbb{Q} \subset (0,1]$ are the rational numbers in $(0,1]$. Note

$$P[X_1 = X_2 = X_3 = 0] = 1$$

and give

$$\sigma(X_i), i = 1, 2, 3.$$

(1)

$$\Omega' = \{0\}, \mathcal{B}' = \sigma(\{0\}) = \{\emptyset, \{0\}\}$$

$$X_1^{-1}(\emptyset) = \emptyset, X_1^{-1}(\{0\}) = \Omega,$$

$$\sigma(X_1) = \{X_1^{-1}(B) : B \in \mathcal{B}'\} = \{\emptyset, \Omega\} = \{\emptyset, (0,1]\}.$$

(2)

$$\Omega' = \{0,1\}, \mathcal{B}' = \sigma(\{0,1\}) = \{\emptyset, \{0,1\}\,\{0\}, \{1\}\}$$

$$X_2^{-1}(\emptyset) = \emptyset, X_2^{-1}(\{0,1\}) = \Omega, X_2^{-1}(\{0\}) = \Omega \backslash \{1/2\}, X_2^{-1}(\{1\}) = \{1/2\}$$

$$\sigma(X_2) = \{X_2^{-1}(B) : B \in \mathcal{B}'\} = \{\emptyset, \Omega, \Omega \backslash \{1/2\}, \{1/2\}\}.$$

(3)

$$\Omega' = \{0,1\}, \mathcal{B}' = \sigma(\{0,1\}) = \{\emptyset, \{0,1\}\,\{0\}, \{1\}\}$$

$$X_3^{-1}(\emptyset) = \emptyset, X_3^{-1}(\{0,1\}) = \Omega, X_3^{-1}(\{0\}) = \mathbb{Q}^c, X_3^{-1}(\{1\}) = \mathbb{Q}$$

$$\sigma(X_3) = \{X_3^{-1}(B) : B \in \mathcal{B}'\} = \{\emptyset, \Omega, \mathbb{Q}^c, \mathbb{Q}\}.$$

3.(2') Suppose $f : \mathbb{R}^k \mapsto \mathbb{R}$, and $f \in \mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$. Let $X_1, ..., X_k$ be random variables on $(\Omega, \mathcal{B})$. Then

$$f(X_1, ..., X_k) \in \sigma(X_1, ..., X_k).$$

Proof:

Let $\mathbb{X} = (X_1, ..., X_k)$, then $\mathbb{X}$ is a random vector $(\Omega, \mathcal{B}) \mapsto (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ since $X_1, ..., X_k$ are random variables. So $\sigma(\mathbb{X}) = \mathbb{X}^{-1}(\mathcal{B}(\mathbb{R}^k)) \subset \mathcal{B}$.

Since $f \in \mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$, $f(\mathbb{X})$ is a random variable $(\Omega, \mathcal{B}) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and $\sigma(f(\mathbb{X})) = (f \circ \mathbb{X})^{-1}(\mathcal{B}(\mathbb{R})) = \mathbb{X}^{-1}(f^{-1}(\mathcal{B}(\mathbb{R})))$.

At the same time, since $f \in \mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$, f itself is a random variable, and $\sigma(f) = f^{-1}(\mathcal{B}(\mathbb{R})) \subset \mathcal{B}(\mathbb{R}^k)$.

Therefore, $\mathbb{X}^{-1}(f^{-1}(\mathcal{B}(\mathbb{R}))) \subset \mathbb{X}^{-1}(\mathcal{B}(\mathbb{R}^k))$. So, $f(X_1, ..., X_k) = f(\mathbb{X}) \in \sigma(\mathbb{X}) = \sigma(X_1, ..., X_k)$.

8.(3') Let X and Y be random variables and let $A \in \mathcal{B}$. Prove that the function

$$Z(\omega) = \begin{cases} X(\omega), & \text{if } \omega \in A \\ Y(\omega), & \text{if } \omega \in A^c \end{cases}$$

is a random variable.

Proof:

It's easy to show that: if X and Y are random variables, then XY and

$X+Y$ are also random variables. Now 1_A and 1_{A^c} are random variables, and $Z = X1_A + Y1_{A^c}$.

9.(3') Suppose that $\{B_n, n \geq 1\}$ is a countable partition of Ω and define $\mathcal{B} = \sigma(B_n, n \geq 1)$. Show a function $X : \Omega \mapsto (-\infty, \infty]$ is $\mathcal{B}$-measurable iff X is of the form

$$X = \sum_{i=1}^{\infty} c_i 1_{B_i},$$

for constants $\{c_i\}$.(What is $\mathcal{B}$?)

Proof:

"$\Leftarrow$"

Note that X is the limit of the summation of random variables $c_i 1_{B_i}$, and in each component, both c_i and 1_{B_i} are $\mathcal{B}$-measurable, therefore every $c_i 1_{B_i}$ is $\mathcal{B}$-measurable, hence X is $\mathcal{B}$-measurable.

Or,

$$\begin{aligned} \forall (a,b] \subset \mathbb{R}, X^{-1}((a,b]) &= X^{-1}(\{c_i : \sum c_i 1_{B_i} \in (a,b]\}) \\ &= \cup (B_i : \sum c_i 1_{B_i} \in (a,b]) \in \sigma(B_n). \end{aligned}$$

"$\Rightarrow$"

By contradiction. Suppose X is not of this form, then

$$\begin{aligned} \exists B_i = B_{i1} + B_{i2}, s.t. X(\omega) > c_i, \forall \omega \in B_{i2} \\ X(\omega) \leq c_i, \forall \omega \in B_{i1} \end{aligned}$$

Then, $\exists (c_i, c_i'), X^{-1}(c_i, c_i') = B_{i2} \notin \sigma(B_n)$, therefore X is not $\mathcal{B}$-measurable.

12.(3') Show that a monotone real function is measurable.

Proof:

Let $f : (a, b] \to (c, d], (a, b, c, d \in \mathbb{R}$ and can be $\pm\infty)$ be monotone, then inverse f exists and is unique.

And it's easy to show that $\forall (c_i, d_i] \subset (c, d], f^{-1}((c_i, d_i]) = (a_i, b_i] \in \mathcal{B}((a, b])$.

Therefore, $\forall C = \cup(c_i, d_i] \in \mathcal{B}((c, d])$,

$f^{-1}(C) = f^{-1}(\cup(c_i, d_i]) = \cup f^{-1}((c_i, d_i]) = \cup(a_i, b_i] \in \mathcal{B}((a, b])$, hence f is $\mathcal{B}((a, b])$-measurable.

22.(3') Suppose $\{X_n, n \geq 1\}$ are random variables on the probability space $(\Omega, \mathcal{B}, P)$ and define the induced random walk by

$$S_0 = 0, S_n = \sum_{i=1}^{n} X_i, n \geq 1.$$

Let

$$\tau := inf\{n > 0 : S_n > 0\}$$

be the first updoing ladder time. Prove τ is a random variable. Assume we know $\tau(\omega) < \infty$ for all $\omega \in \Omega$. Prove S_τ is a random variable.

Proof:

(1) Let $S = (S_0, S_1, ..., S_n)$, then $S : (\Omega, \mathcal{B}) \mapsto (\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$ is measurable.

Now $\tau : (\Omega, \mathcal{B}) \to (\mathbb{Z}^+, 2^{\mathbb{Z}^+})$, and $\forall \{k\} \in 2^{\mathbb{Z}^+}, \tau^{-1}(\{k\}) = \{\omega : \tau(\omega) = k\} = \{\omega : S_0, S_1, ..., S_{k-1} \leq 0, S_k > 0\} \in \mathcal{B}$ since S is measurable.

So τ is a random variable.

(2) Note $S_\tau : (\Omega, \mathcal{B}) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and $\forall (a,b] \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned} S_\tau^{-1}((a,b]) &= \{\omega : S_\tau(\omega) \in (a,b]\} \\ &= \sum_k \{\omega : S_\tau(\omega) \in (a,b]\} \cap \{\omega : \tau(\omega) = k\} \\ &= \sum_k \{\omega : S_k(\omega) \in (a,b]\} \cap \{\omega : \tau(\omega) = k\}. \end{aligned}$$

Now $\{\omega : S_k(\omega) \in (a,b]\} \in \mathcal{B}$ since S_k is measurable, and $\{\omega : \tau(\omega) = k\} \in \mathcal{B}$ since τ is measurable, so $S_\tau^{-1}((a,b]) \in \mathcal{B}$, therefore S_τ is a random variable. Or, you may use a one line proof: $S_\tau = S_k 1_{[\tau = k]}$.

23.(3') Suppose $\{X_1, ...X_n\}$ are random variables on the probability space $(\Omega, \mathcal{B}, P)$ such that

$$P[Ties] := P\{\cup_{i \neq j, 1 \leq i,j \leq n} [X_i = X_j]\} = 0.$$

Define the relative rank R_n of X_n among $\{X_1, ..., X_n\}$ to be

$$R_n = \begin{cases} \sum_{i=1}^n 1_{[X_i \geq X_n]}, & \text{on } [Ties]^c, \\ 17, & \text{on } [Ties]. \end{cases}$$

Prove R_n is a random variable.

Proof:

$R_n : (\Omega, \mathcal{B}(\Omega)) \to (\Omega', \mathcal{B}')$ where $\Omega' = \{1, ..., n\} \cup \{17\}, \mathcal{B}' = \mathcal{B}(\Omega')$.

$\forall \{k\} \in \mathcal{B}', R_n^{-1}(\{k\}) = \{[Ties] \cap [k = 17]\} \cup \{[Ties]^c \cap [\sum_{i=1}^n 1_{[X_i \geq X_n]}] = k]\}$, and every component here is measurable.

Or, you may use a one line proof: $R_n = [\sum_{i=1}^n 1_{[X_i \geq X_n]}] 1_{[Ties]^c} + 17 \cdot 1_{[Ties]}$.