Stat581 HW5 Solutions

1.(3') Let $B_1, ..., B_n$ be independent events. Show

$$P(\bigcup_{i=1}^{n} B_i) = 1 - \prod_{i=1}^{n}(1 - P(B_i)).$$

Proof:

$$\begin{aligned} P(\bigcap_{i=1}^{n} B_i) = \prod_{i=1}^{n} P(B_i) \quad &\Rightarrow \quad P(\bigcap_{i=1}^{n} B_i^c) = \prod_{i=1}^{n} P(B_i^c) \\ &\Rightarrow \quad P([\bigcup_{i=1}^{n} B_i]^c) = \prod_{i=1}^{n}(1 - P(B_i)) \\ &\Rightarrow \quad P(\bigcup_{i=1}^{n} B_i) = 1 - \prod_{i=1}^{n}(1 - P(B_i)) \end{aligned}$$

Note, $B_1, ..., B_n$ are independent implies that $B_1^c, ..., B_n^c$ are also independent. You can prove this by letting $\mathcal{B}_i = \{\emptyset, B_i\}$, and prove $\mathcal{B}_i$ are independent, by basic criterion, so are $\sigma(\mathcal{B}_i)$, and therefore B_i^c are independent. Or simply use inclusion-exclusion formula to prove.

2.(2') What is the minimum number of points a sample space must contain in order that there exist n independent events $B_1, ..., B_n$, none of which has probability zero or one?

Clearly, in order that there exist n independent events $B_1, ..., B_n$, none of which has probability zero or one, there must be at least 2^n points in the sample space, each one belongs to a single partition generated by $B_1, ..., B_n$.

Note, these *strict* inequalities hold:

$$0 = P(\emptyset) < P(\bigcap_{k=1}^{n} B_k) < ... < P(B_i \cap B_j) < P(B_i) < P(\Omega) = 1, \forall i \in 1, ..., n$$

4.(3') Suppose $(\Omega, \mathcal{B}, P)$ is the uniform probability space; that is, $([0,1], \mathcal{B}, \lambda)$ where λ is the uniform probability distribution. Define

$$X(\omega) = \omega.$$

(a) Does there exist a bounded random variable that is both independent of X and not constant almost surely?

No. Suppose Y is a bounded random variable. Since $X(\omega) = \omega$, $Y(\omega) = Y(X(\omega))$, therefore Y is a function of X, hence Y is X-measurable, i.e., $\sigma(Y) \subset \sigma(X)$.

Suppose $Y \perp\!\!\!\perp X$, $\forall A \in \sigma(Y) \subset \sigma(X), P(A) = P(AA) = P(A)^2$, so $P(A) = 0$ or 1, therefore $\exists c \in [0,1], s.t. F_Y(y) = \left\{ \begin{array}{ll} 0, & y < c \\ 1, & y \geq c \end{array} \right.$, i.e., $Y \overset{a.s.}{=} c$

(b) Define $Y = X(1-X)$. Construct a random variable Z which is not almost surely constant and such that Z and Y are independent.

Let $Z = \left\{ \begin{array}{ll} 0, & \omega \in [0, 1/2] \\ 1, & \omega \in (1/2, 1] \end{array} \right.$, clearly, Z is not almost surely constant. Now,

$$\begin{aligned} P(Y \leq y, Z \leq z) &= P(\{\omega : \omega(1-\omega) \leq y\} \cap \{\omega : Z(\omega) \leq z\}), \\ &= \left\{ \begin{array}{ll} \frac{1}{2}P(Y \leq y), & z = 0 \\ P(Y \leq y), & z = 1 \end{array} \right. \\ &= P(Y \leq y)P(Z \leq z) \end{aligned}$$

So, $Y \perp\!\!\!\perp Z$.

5.(b)(2') Suppose X is a random variable. If there exists a measurable

$$g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

such that X and $g(X)$ are independent, then prove there exists $c \in \mathbb{R}$ such that

$$P[g(X) = c] = 1.$$

Proof:

Since $\sigma(g(X)) \subset \sigma(X)$, and $g(X) \perp\!\!\!\perp X$, from problem4, we know $\forall A \in \sigma(g(X)), P(A) = 0$ or 1, and since $g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, therefore $\exists c \in \mathbb{R}, s.t., P[g(X) = c] = 1$. Here, $c = sup\{x : F_{g(X)}(x) = P(g(X) \leq x) = 0\}$.

7.(3') If A, B, C are independent events, show directly that both $A \cup B$ and $A \setminus B$ are independent of C.

Proof:

Either by inclusion-exclusion, or by basic criterion.

Note easy to show that $\{A, B, A \cap B\} \perp\!\!\!\perp C$, therefore $\sigma(A, B) \perp\!\!\!\perp C \Rightarrow \{A \cup B\} \perp\!\!\!\perp C$ and $\{A \setminus B\} \perp\!\!\!\perp C$.

8.(2') If X and Y are independent random variables and f, g are measurable and real valued, why are $f(X)$ and $g(Y)$ independent?

Because $f(X) \in \sigma(X)$ and $g(Y) \in \sigma(Y)$.

9.(3') Suppose $\{A_n\}$ are independent events satisfying $P(A_n) < 1$, for all n. Show

$$P(\bigcup_{n=1}^{\infty} A_n) = 1 \text{ iff } P(A_n \text{ i.o. }) = 1.$$

Give an example to show that the condition $P(A_n) < 1$ cannot be dropped.

Proof:

$\Leftarrow$

$$\begin{aligned} P(A_n \text{ i.o. }) = 1 &\Rightarrow P(\limsup A_n) = 1 \\ &\Rightarrow \lim_{n\to\infty} P(\bigcup_{k=n}^{\infty} A_k) = 1 \\ &\Rightarrow P(\bigcup_{k=1}^{\infty} A_k) = 1 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} P[\bigcup_{n=1}^{\infty} A_n] = 1 &\Rightarrow P(\bigcap_{n=1}^{\infty} A_n^c) = 0 \\ &\Rightarrow \prod_{k=1}^{n} P(A_k^c) \prod_{k=n+1}^{\infty} P(A_k^c) = 0 \\ &\Rightarrow P(\bigcap_{k=n+1}^{\infty} A_k^c) = 0 \\ &\Rightarrow P(\liminf A_n^c) = 0 \\ &\Rightarrow P(\limsup A_n) = 1 \\ &\Rightarrow P(A_n \text{ i.o. }) = 1 \end{aligned}$$

Example: $A_1 = \Omega$ and $A_i = \emptyset, \forall i \neq 1$.

13.(3') Let $\{X_n, n \geq 1\}$ be iid with $P[X_1 = 1] = p = 1 - P[X_1 = 0]$. What is the probability that the pattern 1,0,1 appears infinitely often?

By hint, let $A_k = [X_k = 1, X_{k+1} = 0, X_{k+2} = 1]$, easy to see that A_{3k-2} are independent, and $\sum_k P(A_k) \geq \sum_k P(A_{3k-2}) = \lim_{k\to\infty} kp^2(1-p) = \infty$, so $P(A_k \text{ i.o.}) = 1$.

14.(3') In a sequence of independent Bernoulli random variables $\{X_n, n \geq 1\}$ with

$$P[X_n = 1] = p = 1 - P[X_n = 0],$$

Let A_n be the event that a run of n consective 1's occurs between the 2^n and 2^{n+1}st trial. If $p \geq 1/2$, then there is probability 1 that infinitely many A_n occur.

Proof: Easy to see that A_n are independent. And $P(A_n^c) \leq (1-p^n)^{2^n-n+1} \leq e^{-(2p)^n+(n-1)p^n}$, so $P(A_n) \geq 1-e^{-(2p)^n+(n-1)p^n}$, now note that $\sum_n e^{-(2^n-n+1)p^n}$ converges if $e^{-(2^{n+1}-n)p^{n+1}+(2^n-n+1)p^n} < 1$, therefore when $p \geq 1/2$, $\sum_n P(A_n) \to \infty$ as $n \to \infty$. By Borel 0-1 Law, $P(A_n \text{ i.o.}) = 1$.