Stat581 HW6 Solutions

1.(2') Consider the triangle with vertices (-1,0),(1,0),(0,1) and suppose (X_1, X_2) is a random vector uniformly distributed on this triangle. Compute $E(X_1 + X_2)$.

$$
\begin{aligned}
E(X_1 + X_2) &= \int (X_1 + X_2) P(d\omega) \\
&= \int_S (x_1 + x_2) f(x_1, x_2) d(x_1 \times x_2) \\
&= \int_0^1 dx_2 \int_{x_2 - 1}^{1 - x_2} (x_1 + x_2) dx_1 \\
&= 1/3
\end{aligned}
$$

2.(1') Argue without a computation that if $X \in L_2$ and $c \in \mathbb{R}$, then $Var(c) = 0$ and $Var(X + c) = Var(X)$.

Since $Var(X) = E[(X - E(X))^2]$, the two equations $Var(c) = 0$ and $Var(X + c) = Var(X)$ can easily be explained by properties of expectation.

3.(3') Refer to Renyi's theorem 4.3.1 in Chapter 4. Let

$$L_1 := inf\{j \geq 2 : X_j \text{ is a record.}\}$$

Check $E(L_1) = \infty$.

Proof:

$$E(L_1) = \sum_{i \geq 2} iP(L_1 = i) = \sum_{i \geq 2} \prod_{k=2}^{i-1} (1 - \frac{1}{k}) \text{ diverges.}$$

4.)(2') Let (X, Y) be uniformly distributed on the discrete points (-1,0),(1,0),(0,1),(0,-1). Verify X, Y are not independent but $E(XY) = E(X)E(Y)$.

Proof:

$$P(X = 0, Y = 1) = 1/4 \neq P(X = 0)P(Y = 1) = 1/2 \times 1/4 = 1/8$$

$$E(XY) = E(X) = E(Y) = E(X)E(Y) = 0$$

Note, $\perp\!\!\!\perp \Rightarrow \perp$ but not vice versa.

5(c).(2') If X, Y are random variables with distribution function F, G which have no common discontinuities, then

$$E(F(Y)) + E(G(X)) = 1.$$

Interpret the sum of expectations on the left as a probability.

Proof:

$$\begin{aligned} E(F(Y)) + E(G(X)) &= \int F(y)G(dy) + \int G(x)F(dx) \\ &= \int P(X \le y)G(dy) + \int P(Y \le x)F(dx) \\ &= E(P(X \le Y)) + E(P(Y \le X)) \\ &= 1. \end{aligned}$$

Or, write as $\int\int 1_{x \le y} F(dx)G(dy) + \int\int 1_{y \le x} G(dy)F(dx)$ and by Fubini.

6.(4') Suppose $X \in L_1$ and A and A_n are events.

(a) Show

$$\int_{[|X|>n]} X dP \to 0.$$

Proof: By DCT(or MCT, replacing the bounded condition with a monotonicity condition),

$$\left.\begin{array}{l} X(1_{[|X|>n]}) \to 0 \text{ pointwisely} \\ X(1_{[|X|>n]}) \leq X \in L_1 \end{array}\right\} \Rightarrow E[X(1_{[|X|>n]})] \to 0 \Rightarrow \int_{[|X|>n]} XdP \to 0.$$

(b) Show that if $P(A_n) \to 0$, then

$$\int_{A_n} XdP \to 0.$$

Proof:

$$\begin{aligned} \int_{A_n} |X|dP &= \int_{A_n \cap [|X| \leq M]} |X|dP + \int_{A_n \cap [|X|>M]} |X|dP \\ &\leq \int_{A_n \cap [|X| \leq M]} MdP + \int_{[|X|>M]} |X|dP \\ &\leq MP(A_n) + \int_{[|X|>M]} |X|dP \\ &\to 0, (\text{note, we can always find M large enough when } n \to \infty) \end{aligned}$$

Therefore, $\int_{A_n} XdP \leq \int_{A_n} |X|dP \to 0$.

(c) Show

$$\int_A |X|dP = 0 \text{ iff } P(A \cap [|X| > 0]) = 0.$$

Proof:

$\Rightarrow$

$$\begin{aligned} \int_A |X|dP = 0 &\Rightarrow \int_{A \cap [|X|>0]} |X|dP + \int_{A \cap [|X|=0]} |X|dP = 0 \\ &\Rightarrow \int_{A \cap [|X|>0]} |X|dP = 0 \\ &\Rightarrow \int_{A \cap [|X|>0]} \epsilon dP \leq 0, \exists \epsilon > 0 \\ &\Rightarrow \epsilon P(A \cap [|X| > 0]) \leq 0 \\ &\Rightarrow P(A \cap [|X| > 0]) = 0. \end{aligned}$$

$\Leftarrow$

$$\int_A |X|dP = \int_{A\cap[|X|>0]} |X|dP + \int_{A\cap[|X|=0]} |X|dP = \int_{A\cap[|X|>0]} |X|dP = 0,$$

the last equality is from (b), consider $A_n = A \cap [|X| > 0]$.

10.(3') For $X \geq 0$, let

$$X_n^* = \sum_{k=1}^{\infty} \frac{k}{2^n} 1_{[\frac{k-1}{2^n} \leq X \leq \frac{k}{2^n}]} + \infty 1_{[X=\infty]}.$$

Show

$$E(X_n^*) \downarrow E(X).$$

Proof:

Note that $X_n^* \downarrow X$, so we want to use MCT.

(1) If $X < \infty$, i.e., $X < C$, then $X_n^*(\omega) = \sum_{k=1}^{\ulcorner C2^n \urcorner} \frac{k}{2^n} 1_{[\frac{k-1}{2^n} \leq X(\omega) \leq \frac{k}{2^n}]}$, and $\forall \omega, X_n^*(\omega) - X(\omega) \leq 2^{-n} \downarrow 0$, also, note that $X_n^* \in L_1$ since $X_n^* \leq X + 1$. Therefore, MCT applies, $E(X_n^*) \downarrow E(X)$.

(2) If $X = \infty$, then $X_n^* = \infty$, so $E(X_n^*) \downarrow E(X)$.

15.(3') Suppose X is a non-negtive random variable satisfying

$$P(0 \leq X < \infty) = 1.$$

Show

$$(a) \quad \lim_{n\to\infty} nE(\frac{1}{X} 1_{[X>n]}) = 0,$$
$$(b) \quad \lim_{n\to\infty} n^{-1}E(\frac{1}{X} 1_{[X>n^{-1}]}) = 0.$$

Proof:

(a)

$$\lim_{n\to\infty} nE(\frac{1}{X}1_{[X>n]}) \leq \lim_{n\to\infty} E(1_{[X>n]}) = \lim_{n\to\infty} P(X > n) = 0.$$

(b)

$$\begin{aligned} n^{-1}E(\frac{1}{X}1_{[X>n^{-1}]}) &= E(\frac{1}{nX}1_{[\frac{1}{nX}<1]}) \\ &= E(\frac{1}{nX}1_{[0<\frac{1}{nX}\leq M]}) + E(\frac{1}{nX}1_{[M<\frac{1}{nX}<1]}) \\ &\leq M + P(M < \frac{1}{nX} < 1) \end{aligned}$$

Now, choose M to be small enough, then first term $\to 0$, and for fixed M, second term also $\to 0$ as $n \to \infty$.