STRUCTURES

Algebras and $\sigma-$algebras

Set operations and closure

Set class (family) is *closed* under set operation if the result of the operation on sets from this class also belongs to this class.

Countable union

$$\cup_{j=1}^{\infty} A_j$$

What classes (families) of sets might be closed under countable union?

1. Consider set Ω its power set 2^{Ω}. 2^{Ω} is closed under countable union since *any* union of subsets of Ω is a subset of Ω.

2. Consider the family of finite unions of intervals of the form $[a, b)$, where $a, b \in R$. This family is not closed under countable union.

Arbitrary union

$$\cup_{t\in T} A_t$$

1. Discrete sets (sets of isolated points) are usually closed under countable union, but not closed if $\#\{T\} = \mathfrak{C}$.

2. More generally, *open sets* are closed under arbitrary union (by definition), but *closed sets* not necessarily (these latter are closed under arbitrary intersection).

Definition 1.5.2. A field (=algebra) $\mathcal{A}$ is a non-empty class of subsets of Ω closed under finite union, intersection, and complementation.
Minimum set of postulates

1. $\Omega \in \mathcal{A}$
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

Some comments ...

Definition 1.5.3. A $\sigma-$field ($\sigma-$algebra) $\mathcal{B}$ is a non-empty class of subsets of Ω closed under countable union, countable intersection, and complementation. Minimum set of postulates

1. $\Omega \in \mathcal{B}$
2. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$
3. $B_1, B_2, \ldots, B_i, \ldots \in \mathcal{B} \Rightarrow \cup_{i=1}^{\infty} B_i \in \mathcal{B}$

Examples: Power set, trivial $\sigma-$field, countable/co-countable...

Consider the family of finite unions of intervals of the form $[a, b)$, where $a, b \in (0, 1]$. It is a field but not a $\sigma-$field.

Minimal $\sigma-$fields = $\sigma-$fields generated by a class of sets

Theorem (w/o proof). Family of families $\{C_t, t \in T\}$ such that each family C_t closed under operation O. Then

$$C = \cap_{t \in T} C_t \text{ is closed under } O$$

$\Rightarrow$ Arbitrary intersection of $\sigma-$fields is a $\sigma-$field!

Proposition 1.6.1. There exists a minimal $\sigma-$field $\sigma(C)$ containing a given class C of subsets of Ω.

Proof

$$\begin{aligned} \mathfrak{K} &= \{\mathcal{B} : \mathcal{B} \text{ is a } \sigma - field, C \subseteq \mathcal{B}\} \\ &\Rightarrow \text{ and } \cap_{\mathcal{B}\in\mathfrak{K}} \mathcal{B} \text{ is a } \sigma - field \end{aligned}$$

Is $\cap_{\mathcal{B}\in\mathfrak{K}}\mathcal{B} = \sigma(C)$?

Suppose $\exists\sigma'$ (a $\sigma - field$):

$$C \subseteq \sigma' \subseteq \cap_{\mathcal{B}\in\mathfrak{K}}\mathcal{B}$$

But $C \subseteq \sigma' \Rightarrow \sigma' \in \mathfrak{K} \Rightarrow \cap_{\mathcal{B}\in\mathfrak{K}}\mathcal{B} \subseteq \sigma' \Rightarrow \cap_{\mathcal{B}\in\mathfrak{K}}\mathcal{B} = \sigma'$

$\square$

Borel sets on the real line

Suppose $\Omega = \mathbb{R}$

$$C = \{(a, b], -\infty \leq a \leq b < \infty\}$$

Define "the Borel sets of $\mathbb{R}$" as

$$\mathcal{B}(\mathbb{R}) := \sigma(C)$$

"Obvious properties"

$$\begin{aligned}
\mathcal{B}(\mathbb{R}) &= \sigma\{(a,b], -\infty \leq a \leq b < \infty\} = \sigma(C^{(]}) \\
&= \sigma\{[a,b], -\infty \leq a \leq b < \infty\} = \sigma(C^{[]}) \\
&= \sigma\{[a,b), -\infty \leq a \leq b < \infty\} = \sigma(C^{[)}) \\
&= \sigma\{(a,b), -\infty \leq a \leq b < \infty\} = \sigma(C^{()})
\end{aligned}$$

give a sample proof ...

Also,

$$\mathcal{B}(\mathbb{R}) := \sigma(C^{()}) = \sigma(\text{ open sets })$$

Proof is based on the following property

$$\begin{aligned} & O \text{ is an open set on } \mathbb{R} \\ \Longrightarrow \quad & O = \cup_{j=1}^{\infty} I_j, \text{ where } \{I_j\} \text{ are disjoint open intervals} \end{aligned}$$

Therefore,

$$\begin{aligned} I_j \quad &\in \quad C^{(\,)} \subset \sigma(C^{(\,)}) \Longrightarrow \forall O = \cup_{j=1}^{\infty} I_j \in \sigma(C^{(\,)}) \\ &\Longrightarrow \quad \sigma(\text{ open sets }) \subseteq \sigma(C^{(\,)}) \end{aligned}$$

and on the other hand

$$C^{(\,)} \subseteq \{\text{open sets}\} \Longrightarrow \sigma(C^{(\,)}) \subseteq \sigma(\text{ open sets })$$

□

Comparing Borel sets (self-study)

Suppose $\Omega_0 \subset \Omega$; define a restriction of class C to subspace Ω_0

$$C \cap \Omega_0 := C_0 = \{A \cap \Omega_0 : A \in C\}$$

Theorem

$$\sigma(C_0) = \sigma(C) \cap \Omega_0$$