Mathematical Probability 1: Stat 581
Fall 2006

Test 1

Closed book, closed notes, 3 hours, cheat sheet as agreed upon
Numbers in square brackets are maximum points per problem

1. Consider an infinite (denumerable) sequence of Bernoulli trials with probability of success equal to p in a single trial. Demonstrate that event $A = \{\text{success ever happens}\}$ has probability 1. Hint: Consider events $A_i = \{\text{first success happens at trial } i\}$. [5]

2. Consider two random variables X and Y defined on the same probability space. Suppose that $X(\omega) \geq Y(\omega)$, for all $\omega \in \Omega$. Demonsstrate that the distribution functions satisfy $F_X(x) \leq F_Y(x)$, for all $x \in \mathbb{R}$. [5]

3. Show that the indicator function of a measurable set is measurable. [5].

4. Let X be a random variables on $(\Omega, \mathcal{B})$ and let $A \in \mathcal{B}$. Prove that

$$Z(\omega) = \begin{cases} X(\omega) & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

is a random variable. [7]

5. Define σ-algebra, π-system and λ-system. Demonstrate that a structure that is at the same time a π-system and λ-system is a σ-algebra. [10]

6. Find the sigma-algebra $\sigma(\mathcal{C})$ generated by a family of subsets of $\Omega = [0, 1]$, where $\mathcal{C} = \{[0, .3], [.3, 1], \emptyset\}$. [5]

7. Suppose $\{A_n, n \geq 1\}$ are events (measureable subsets of Ω). Suppose that

$$P(\Omega \setminus \bigcup_{n \geq 1} A_n) = 1.$$

Show that $P(A_n) = 0$ for each n. [5]

8. Suppose $\{A_n, n \geq 1\}$ are sets such that $A_n \supset A_{n+1}$. Demonstrate that $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \cap_{n\geq 1} A_n$. Show that indicator functions of sets A_n tend to the indicator function of set A. [5]

9. Suppose we consider measurable space $([0,1], \mathcal{B}[0,1])$. Let $X(\omega)$ equal to 1 for ω rational and to 0 for ω not rational. Is $X(\omega)$ a random variable? [5]