Statistics 310 / Econ 382
Final Exam
Spring 2006

- The exam is due on May 10th by 5:00 pm. If you are a graduating senior please turn in your exam by May 4th @ 5 PM so that I have time to grade your exams and turn in your final grades by the 6th.
- This is a 5 hour exam.
- Please put your exam in a sealed envelope. If you do not find me in my office please put it under the door.
- All honor code rules apply.
- This is an open book, open notes exam. You may consult any book, the internet, or any materials that you like. You MAY NOT, however, consult with anyone else (TAs, faculty, any human being) via email, telephone, etc.
- Please write legibly and organize your work. I cannot give you partial credit if I cannot read what you write.
- Please be as detailed as possible in your answers. You must show all your work to receive full credit.
- There are 11 problems. You must work problems 1 and 2. Of the remaining 9 problems you must attempt at least 6 problems. If you attempt more than 6 specify which ones you want graded. Otherwise the first 6 will be graded.
- Each problem is worth 12.5 points.

Problem 1.- Write, in your own words, about the Central Limit theorem. Your discussion must address the important facts about the CLT and its applications. You must discuss the various assumptions made and how they fail or hold for various distributions. You may want to report results of computer simulations showing the CLT at work for various distributions. If reporting simulations, please include the computer code used for the simulations. (minimum of 400 words not including simulations/code).

Problem 2.- Choose two of the following five topics and discuss, in your own words, the ideas, concepts, and applications associated with the selected topics. (minimum of 200 words/topic).

- Confidence intervals for the difference of two normal populations.
- The law of large numbers.
- Connection between confidence intervals and hypotheses testing.
- Unbiasedness, maximum likelihood estimation, Bayes estimation.
- p-values in testing

Problem 3.- Consider the following systems of identical and independent components. The first system (A) operates if and only if components 1 and 2 and 3 work or if components 4 and 5 and 6 work. System (B) works if and only if 1 or 4 work and 2 or 5 work and 3 or 6 works. Suppose that the probability that a component works is .8. Calculate the probability that each of the systems work.
Problem 4.- Let X be a binomial random variable with n trials and probability of success p.
   • Compute the moment generating function for X.
   • Let Y be also binomial (m, q) and suppose that X and Y are independent.
     Compute the moment generating function for X+Y. Is X+Y a binomial random variable?
   • Calculate the conditional probability that X=k given that X+Y=j.

Problem 5.- A manufacturer of Christmas tree light bulbs knows that 2% of its bulbs are defective. Let X be the number of defective bulbs in a box of 100.
   • Assuming independence, calculate the probability that not more than two defective bulbs are found in a box.
   • Approximate the probability using the Poisson distribution.
   • Approximate the same probability using the Central Limit Theorem. (With and without continuity correction.)

Problem 6.- Let X have density function given by \( f(x) = \lambda e^{-\lambda x} \) where \( x > 0 \) and \( \lambda > 0 \).
   • Compute the following probability \( P(X > x + y | X > x) \) and show that it is equal to \( P(X > y) \)
   • Calculate the mean and variance of X.
   • Compute the moment generating function when \( \lambda = 1 \), and argue that the sum of independent and identically distributed exponential random variables with \( \lambda = 1 \) is not distributed like an exponential random variable.

Problem 7.- Let \( X_1, \ldots, X_n \) represent a random sample from a normal distribution with mean \( \theta \) and variance \( \sigma^2 \), where it is assumed that \( \sigma^2 \) is known. Suppose that a prior
distribution for $\theta$ is given as $\pi(\theta)$ where $\pi(\theta)$ is a normal density with mean $\mu$ and variance 1.

- Compute the posterior distribution $\pi(\theta | X_1, \ldots, X_n)$
- Using this, find the Bayes estimator for $\theta$.
- Is the Bayes estimator an unbiased estimator for $\theta$?

**Problem 8.** Let $(X, Y)$ have joint density function given by

$$f(x, y) = \begin{cases} kxy & \text{if } x, y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find $k$ so that $f(x, y)$ defines a density function.
- Compute the marginal density functions of both $X$ and $Y$.
- Are $X$ and $Y$ independent? Are they uncorrelated? If not uncorrelated, calculate the correlation coefficient.

**Problem 9.** Political candidate A claims to have the advantage over candidate B in the upcoming elections with 54% of the vote against 46%. (Suppose that there are only two candidates in the race and that $p_A + p_B = 1$. Candidate B commissions a statistician to carry out a poll. Candidate B wants to estimate his support with not more than 3% error and with 95% confidence.

- Determine the sample size needed by the statistician to accomplish his job.
- Suppose that it turns out that the poll conducted by candidate A consisted of 400 interviews, that yielded 54% in his/her favor and 46% for candidate B. Let $p_A$ and $p_B$ denote the true proportions in favor of A and B respectively. Is this enough evidence to reject the null hypothesis $H_0 : p_A = p_B$ in favor of the hypothesis $H_A : p_B > p_A$?
- Compute the p-value of the test.

**Problem 10.** Suppose that $X \sim U(0, \theta)$, where $\theta > 0$. Consider testing, at level $\alpha = .05$ the hypothesis $H_0 : \theta = 1$ vs $H_A : \theta > 1$. Given one observation of $X$, consider the test that rejects $H_0$ when $X > k$.

- Find $k$ to achieve the probability of type 1 error equal to .05
- Compute the power of the test for $\theta > 1$. 
Problem 11.- Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_m \) denote independent random samples from populations with distributions \( N(\mu_1, \sigma_1^2) \) and \( N(\mu_2, \sigma_2^2) \). Discuss the distributions of the following random variables:

- \( \bar{X} \) and \( \bar{Y} \)
- \( (n-1)S_X^2 / \sigma_1^2 \) and \( (m-1)S_Y^2 / \sigma_2^2 \)
- \( \sqrt{n} (\bar{X} - \mu_1) / \sigma_1 \) and \( \sqrt{n} (\bar{Y} - \mu_2) / S_Y \)
- \( (n-1)S_X^2 / \sigma_1^2 \) and \( (m-1)S_Y^2 / \sigma_2^2 \)
- \( (m-1)S_Y^2 / \sigma_2^2 \) and \( (n-1)S_X^2 / \sigma_1^2 \)