Problem 1.1 (Coin flips)

A fair coin is flipped until the first head occurs. Let $X$ denote the number of flips required. (a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$
\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \quad \sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}.
$$

(b) A random variable $X$ is drawn according to this distribution. Find an “efficient” sequence of yes-no questions of the form, “Is $X$ contained in the set $S$?”. Compare $H(X)$ to the expected number of questions required to determine $X$.

Problem 1.2 (Entropy of functions of a random variable)

Let $X$ be a discrete random variable. Show that the entropy of a function $g(X)$ is less than or equal to the entropy of $X$ by justifying the following steps:

$$
H(X, g(X)) \overset{(a)}{=} H(X) + H(g(X) | X) \\
\overset{(b)}{=} H(X); \\
H(X, g(X)) \overset{(c)}{=} H(g(X)) + H(X | g(X)) \\
\overset{(d)}{\geq} H(g(X)).
$$

Thus $H(g(X)) \leq H(X)$.

Problem 1.3 (Zero conditional entropy)

Show that if $H(Y | X) = 0$, then $Y$ is a function of $X$, i.e., for all $x$ with $p(x) > 0$, there is only one possible value of $y$ with $p(x, y) > 0$.

Problem 1.4 (World Series)

The World Series is a seven-game series that terminates as soon as either team wins four games. Let $X$ be the random variable that represents the outcome of a World Series between teams A and B; possible values of $X$ are AAAA, BABABAB, and BBBAAAA, etc. Let $Y$ be the number of games played, which range from 4 to 7. Assuming that A and B are equally matched and that the games are independent, calculate $H(X), H(Y), H(Y | X)$, and $H(X | Y)$.
Problem 1.5 (A metric)

A function $\rho(x, y)$ is a metric if for all $x, y$

- $\rho(x, y) \geq 0$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, y) = 0$ if and only if $x = y$
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$

(a) Show that $\rho(X, Y) = H(X|Y) + H(Y|X)$ satisfies the first, second, and fourth properties above. If we say that $X = Y$ if there is a one-to-one function mapping $X$ to $Y$, then the third property is also satisfied, and $\rho(X, Y)$ is a metric.

(b) Verify that $\rho(X, Y)$ can also be expressed as

$$
\rho(X, Y) = H(X) + H(Y) - 2I(X; Y)
= H(X, Y) - I(X; Y)
= 2H(X, Y) - H(X) - H(Y)
$$