1. Short Questions

a. \[
E[V] = \int_0^4 \frac{4}{3} \pi r^3 f(r) \, dr \\
= \int_0^4 \frac{\pi}{6} r^4 \, dr \\
= \frac{\pi}{6} \left[ \frac{r^5}{5} \right]_0^4 \\
= \frac{512\pi}{15} \approx 107.
\]

Some people computed \( E[V] = \frac{4}{3} \pi E[R^3] \), but keep in mind it is generally not true that \( E[g(X)] = g(E[X]) \) unless \( g \) is linear.

b. Let \( X \) be the total number of errors in the sequence of \( n \) information bits, and let 
\[
X_i = \begin{cases} 
1 & \text{if the } i\text{th information bit is decoded incorrectly} \\
0 & \text{otherwise}
\end{cases}
\]
so that \( X = \sum_{i=1}^n X_i \). Denote \( \tilde{p} = P(X_i = 1) \), so that \( X_i \sim \text{Ber}(\tilde{p}) \) and \( X \sim \text{bin}(n, \tilde{p}) \). Then 
\[
P(\text{an error}) = 1 - P(\text{no error}) \\
= 1 - P(X = 0) \\
= 1 - (1 - \tilde{p})^n,
\]
so it remains to find \( \tilde{p} \). Note that \( X_i = 1 \) if and only if 3 or more of the five repeated bits are corrupted. Thus 
\[
\tilde{p} = P(3, 4, \text{ or 5 flipped bits out of 5}) \\
= \binom{5}{3} p^3 (1 - p)^2 + \binom{5}{4} p^4 (1 - p) + \binom{5}{5} p^5 \\
= 0.0579
\]
where I have plugged in \( p = 0.2 \). In conclusion, \( P(\text{an error}) = 1 - (0.94)^n \).

c. Let \( X \) be the point where the stick is broken. Then \( X \sim \text{unif}[0, 1] \). The probability of interest is 
\[
P(X \in [0, 1/3] \cup [2/3, 1]) = F(1/3) - F(0) + F(1) - F(2/3) \\
= 1/3 + 1/3 \\
= 2/3
\]
where \( F(x) = x \) is the CDF of \( X \). Note: Many people looked at the random variable

\[ Y = \max(X, 1 - X), \]
the length of the longest piece, and computed \( P(Y \geq 2/3) \). This is fine in principle. However, you cannot simply say that \( Y \sim \text{unif}[1/2, 1] \) without any justification. While true, this must shown by, for example, finding the CDF of \( Y \) and identifying it as that of a uniform.
\[ d. \]

\[
P(1 \leq Y \leq 3) = P(\ln 1 \leq X \leq \ln 3)
\]
\[
= F_X(\ln 3) - F_X(\ln 1)
\]
\[
= \Phi\left(\frac{\ln 3 - 1}{\sqrt{2}}\right) - \Phi\left(\frac{\ln 1 - 1}{\sqrt{2}}\right)
\]
\[
= \Phi(0.0679) - \Phi(-0.7071)
\]
\[
= \Phi(0.0679) - (1 - \Phi(0.7071))
\]
\[
= 0.528 - (1 - 0.761)
\]
\[
= 0.289
\]

2. Poker hands

a.

\[
\binom{49}{2} = 1176
\]

b. One pair?

\[
\frac{\binom{44}{2} - 11 \binom{4}{2}}{\binom{49}{2}} = 0.748
\]

where the subtraction is to discount the possibility of drawing a new pair, which would boost the hand to two pairs.

c. Two pair?

\[
\frac{\binom{3}{1} \binom{44}{1} + 11 \binom{4}{2}}{\binom{49}{2}} = 0.168
\]

where the first term on top accounts for drawing another jack, and the other term on top for drawing a new pair. Remember, drawing a 3 would bump the hand up in status.

d. Three of a kind?

\[
\frac{\binom{3}{1} \binom{44}{1}}{\binom{49}{2}} = 0.075
\]

where the number on top is the number of ways to draw one three and one non-three/jack. Drawing two jacks would yield a full house.

3. Median Absolute Deviation

a.

\[
E[|X - \mu|] = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{2a} \, dx
\]
\[
= 2 \int_{0}^{a} \frac{x}{2a} \, dx
\]
\[
= \frac{1}{a} \int_{0}^{a} x \, dx
\]
\[
= \frac{1}{a} \left[ \frac{1}{2} x^2 \right]_{0}^{a}
\]
\[
= \frac{a}{2}
\]
By comparison, the standard deviation is

\[
\sqrt{\frac{(2a)^2}{12}} = \frac{a}{\sqrt{3}} > \text{MAD}(X).
\]

b.

\[
E[|X - \mu|] = \int_{-\infty}^{\infty} |x| f(x) \, dx
= 2 \int_0^{\infty} x f(x) \, dx
= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} xe^{-\frac{x^2}{2\sigma^2}} \, dx
= \frac{2}{\sqrt{2\pi\sigma^2}} \left[ -\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \right]_0^{\infty}
= \sqrt{\frac{2}{\pi}} \sigma < \sigma = \text{STD}(X)
\]

4. MGF

a.

\[
M(t) = \sum_{x=1}^{\infty} e^{xt} \cdot \frac{1}{\ln 2} \cdot \frac{1}{x^{2t}}
= \frac{1}{\ln 2} \sum_{x=1}^{\infty} \frac{1}{x} \left( \frac{e^t}{2} \right)^x
= \frac{-1}{\ln 2} \ln \left( 1 - \frac{e^t}{2} \right),
\]

for \( t < \ln 2 \).

b.

\[
\mu = M'(0) = \frac{1}{\ln 2}
\]

and

\[
\sigma^2 = M''(0) - \mu^2 = \frac{2}{\ln 2} - \left( \frac{1}{\ln 2} \right)^2
\]