Useful Identities

- $\Phi(z) \equiv \text{erf}$

For the error function and for c.d.f. of Gaussian distribution, using $\Phi(-z) = 1 - \Phi(z)$:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = 2\Phi(\sqrt{2}x) - 1$$

$$\Phi(z) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{z}{\sqrt{2}} \right) \right)$$

- $\int e^{-x^2} \, dx$ and modulus

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi} \quad \int_{-\infty}^{\infty} e^{-x^2/2} = \sqrt{2\pi} \quad \int_{-\infty}^{\infty} e^{-2x^2} = \sqrt{\frac{\pi}{2}}$$

- The Factorial (Gamma) Function and its identities. Note that in full generality the function is in fact a solution to an extended complex integral (a “meromorphic” function); our familiar improper integral is only defined for argument with real part $> 0$.

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt, \quad \text{Re}(\alpha) > 0.$$ 

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) = \alpha! = \alpha(\alpha - 1)!, \quad \text{giving } \Gamma(\alpha) = (\alpha - 1)! \quad \text{and } \Gamma(\alpha + \nu) = (\alpha + \nu - 1)!.$$ 

The Digamma function is $\psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)}$. Recurrence relations include

$$\psi(z + 1) = \psi(z) + \frac{1}{z} \quad \text{and} \quad \psi(1 - z) = \psi(z) + \pi \cot \pi z.$$ 

\[ \psi(z) \] is the trigamma function. The Beta function is related to \( \Gamma(z) \):

\[
B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, dx = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} \, dt, \quad \text{or} \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.
\]

These are normalizing constants used for pdf kernels.

### Landau Order Notation

Landau order notation is a convenient way to summarize remainder terms of sequences or expansions (or tail events when dealing with random phenomena) for use in truncations; it does not give an exact statement of the remainder error, but rather an approximation to this error which may then be used to simplify other calculations and results.

We will consider the case in \( \mathbb{R}^n \); suppose we have sequences of real numbers \( a_n \) and \( b_n \); we will also be considering sequences of random vectors \( X_n, Y_n, \) etc.

### Deterministic Order Notation

### Stochastic Order Notation

### Mathematical Spaces

Space – A set \( X \)

Algebra or Topology

Topological Space

Measureable Space

Probability Space

Linear Space

Metric Space

Vector Space
Normed Linear Space

Inner Product Space

Examples: vector spaces of functions

- $C_b(x)$: Bounded, continuous functions on $\mathcal{X}$

Sets

Lim sup $A_n = [A_n \text{ i.o.}]$

An i.o. == $M_n \geq 1 U_m = n A_m$. No matter how large $n$, there $m \geq n$ and omega in $A_m$.

Lim inf

Outcome eventually there is an $n$ s.t. for any $m \geq n$, omega is in $A_m$. For for this $n$ and up, omega is in $A_n$.

Ex, due to Florescu: toss coin infinitely many times, 1 come up infinitely many times.

Convergence and Continuity

Convergence of sequences

Uniform continuity

$\varepsilon, \delta$, are not functions of $x \; \forall x$.

Convergence of sequence of functions - Pointwise

$f_n, f, g \in$ functional space $\mathcal{X}$, typically a metric, normed linear, or inner product space. W.s. $f_n \xrightarrow{\text{pw}} f$

as $n \to \infty$, or $\lim f_n(t) = f(t)$; i.e., $\forall t \in \mathcal{X}$ and $\forall \varepsilon, n_\varepsilon > 0$, $\|f_n(t) - f(t)\| < \varepsilon \; \forall n > n_\varepsilon$. Procedure: fix $t$ and check the convergence of the sequence of values of $f_n(t)$ to $f(t)$.

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Convergence of sequence of functions - Uniform

Analogous to uniform continuity (which of course we have provided yet), $n_\varepsilon$ is not a function of $x$:

$$\forall n_\varepsilon \neq n_\varepsilon (t) \exists \| f_n (t) - f (t) \| < \varepsilon \ \forall n > n_\varepsilon , \text{ and all } \forall t \in \mathcal{X}.$$  If there is a defined metric $d(f, g)$ then we can say

$$f_n \rightarrow f \iff d(f_n, f) \rightarrow 0.$$  

In a NLS such as $\mathbb{R}^n$, one such metric is the norm-induced metric, or the supremum metric. Since we will normally be concerned with distribution functions, suppose $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$; define

$$d_\varepsilon (f, g) = \sup_{t \in \mathbb{R}^d} \| f(t) - g(t) \| = \sup_{p=1} \| f(t) - g(t) \|.$$  

**Continuity**

Heine’s definition/concept is that a continuous function maps convergent sequences into convergent sequences.

Continuity of a function – pointwise

Continuity of a function - uniform

- Indented stuff
- Bulleted stuff

**Real Analysis**

**Fundamental Theorems**

**FTA**

**Fundamental Theorems of Calculus**\(^3\)

Mean Value Theorem (MVT):

Suppose $g$ is continuous on $[a, b]$, and differentiable on $(a, b)$. Then $\exists$ (at least one)\(^3\)
\[ \lambda \in (a, b) \text{ such that } g'(\lambda) = \frac{g(b) - g(a)}{b - a} \iff g(b) - g(a) = g'(\lambda)(b - a). \]

Leibnitz Rule: for differentiable \( f(x, \theta), a(\theta) \) and \( b(\theta) \):

\[
\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x, \theta) dx = f(b(\theta), \theta) \frac{d}{d\theta} b(\theta) - f(a(\theta), \theta) \frac{d}{d\theta} a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x, \theta) dx
\]

References